QFT Spring2025

HW4

Q1 Muon Decay

- Follow QFT amplitudes class notes to work out the width of muon decay:

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}.$$

Make sure you verify the 3-body integral.

Q2 SU(N) Group Structure

- a) SU(3) Weights and Roots (Follow class notes) Use the eigenvalues of T_3, T_8 ((H_3, H_8)) to label the basis vectors (e_1, e_2, e_3)

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Draw the SU(3) weight triangles in the plane of (H_3, H_8). Construct the 6 vectors which map these weights to one another, e.g. $(1, 0)$ maps e_2 to e_1. Show that

$$T_1 + iT_2$$

is the matrix that does that. What about the other 5's?

- b) Consult the use of Young tableaux and demonstrate for SU(2)

$$2 \otimes 2 = 3 \oplus 1$$
$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$$

and for SU(3)

$$3 \otimes 3 = 6 \oplus \bar{3}$$
$$3 \otimes \bar{3} = 8 \oplus 1$$
$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1.$$

Q3 Cartan Algebra

- a) Consider a general diagonal SU(3) matrix parametrized via

$$\hat{\ell} = \text{diag}\left(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}\right)$$

where $\phi_3 = -\phi_1 - \phi_2$ to ensure $\det \hat{\ell} = 1$.

Construct the Polyakov loop variable via

$$\ell = x + iy = \frac{1}{3} \operatorname{tr} \hat{\ell}.$$

Find an analytic expression for $x(\phi_1, \phi_2)$ and $y(\phi_1, \phi_2)$.

- b) Construct the Jacobian J, i.e.

$$dxdy = J \, d\phi_1 d\phi_2.$$

- c) Compute the determinant of the Vandermonde Matrix

$$V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix}$$

where $\lambda_j = e^{i\phi_j}$.

Verify (after a lot of trigonometry) that

$$|\det V| = 18 \, |J|$$

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Q4 Landau Pole

- Review the notes on polarization. Show that

$$\frac{e^2}{1 - \Pi^{\text{reg}}(q)} = \frac{e_R^2}{1 - \Pi^R(q)}$$

$$\Pi^R(q) = \frac{e_R^2}{2\pi^2} \int_0^1 dx \, x(1-x) \, \ln \frac{m^2 - x(1-x)q^2}{m^2}$$

- For a running coupling constant, consider spacelike momentum $Q^2 = -q^2$

$$\alpha(Q^2) = \frac{\alpha_0}{1 - \Pi^R(q^2 = -Q^2)}$$

$$\alpha_0 = \frac{1}{137}.$$

Derive the large Q^2 limit of $\Pi^R(q^2 = -Q^2)$ and estimate the Landau Pole, i.e. the scale where $\alpha(Q^2)$ diverges.(take $m = 0.5$ MeV)