QFT Spring2025

HW3

Q1 Differential Forms

- a) Verify that $d^2 = 0$ by acting on an arbitrary 0-form and 1-form in 3D.
- b) Fill in the steps to show that

$$\begin{aligned} F &= dA \\ dF &= 0. \end{aligned}$$

  leads to 2 of the Maxwell Equations, i.e. absence of magnetic monopole and Faraday's law.
- c) Unpack the remaining twos:

$$\begin{aligned} d\,{}^*F &= -{}^*j \\ d^*j &= 0. \end{aligned}$$

Q2 Self-energy (part II).

- a) Prove the Feynman parametrization:

$$\frac{1}{AB} = \int_0^1 dx\, \frac{1}{(xA + (1-x)B)^2}.$$

- b) The self energy Σ_R of a resonance is obtained from

$$\Sigma_R = ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q_1^2 - m_1^2 + i\delta} \frac{1}{q_2^2 - m_2^2 + i\delta}$$

  where

$$\begin{aligned} q_1 &= q \\ q_2 &= P - q \\ P^2 &= s = (P^0)^2 - \vec{P}^2 \end{aligned}$$

  are Minkowski 4-vectors. Use the Feynman parametrization to show that

$$\Sigma_R = ig^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{1}{((\tilde{q})^2 - \Delta)^2}$$

  where

$$\begin{aligned} \tilde{q} &= q - (1-x)P \\ \Delta &= xm_1^2 + (1-x)m_2^2 - x(1-x)s - i\delta. \end{aligned}$$

- c) After a shift of integration variable and a Wick's rotation:

$$\begin{aligned} d^4 q &\to i d^4 q_E \\ q^2 &\to -q_E^2 = -(q_4^2 + \vec{q}^2), \end{aligned}$$

  we obtain

$$\Sigma_R = -g^2 \int_0^1 dx \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{(q_E^2 + \Delta)^2}$$

  where q_E is in Euclidean space.

  Use the Schwinger proper time regularization and perform the momentum integral. Show that

$$\Sigma_R = \frac{g^2}{16\pi^2} \int_0^1 dx \ln \Delta + C.$$

  This is the starting point of (HW02, Q3).

  Hint: Recall the Schwinger proper time regularization scheme

$$\begin{aligned} \mathcal{A}^{-1} &= \int_0^\infty dt\, e^{-t\mathcal{A}} \\ \ln \mathcal{A} &= -\int_0^\infty dt\, \frac{1}{t} \left(e^{-t\mathcal{A}} - e^{-t\mathcal{I}} \right). \end{aligned}$$

  The following relation is also useful:

$$\mathcal{A}^{-2} = \int_0^\infty dt\, t\, e^{-t\mathcal{A}}.$$

Q3 Schwinger Proper Time Regularization (part II)

- a) Recall the use of Schwinger proper time regularization (HW02, Q2) to calculate

$$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \frac{1}{p_4^2 + \omega^2} = \frac{1}{2\sqrt{\omega^2}}.$$

  Now consider the divergent integral

$$W[\omega; \Lambda] = \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \ln \left(p_4^2 + \omega^2 \right).$$

  Use the Schwinger proper time technique to regulate the integral, i.e. replace the lower limit of $t \to \frac{1}{\Lambda^2}$, and show that

$$W[\omega;\Lambda] = -\int_{1/\Lambda^2}^{\infty} dt\, \frac{1}{t^{\frac{3}{2}}} \frac{1}{2\sqrt{\pi}}\, e^{-t\omega^2} + C.$$

C is an integration constant.

- b) Study the integral at large Λ. Show that

$$W[\omega;\Lambda] = -\frac{\Lambda}{\sqrt{\pi}} + \omega + \mathcal{O}(1/\Lambda).$$

- c) This suggests the definition of a physical W function:

$$W_{\text{phys.}}[\omega] = \lim_{\Lambda\to\infty} \left(W[\omega;\Lambda] - W[0;\Lambda]\right) = \omega.$$

Re-derive the previous result (again!) via

$$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \frac{1}{p_4^2+\omega^2} = \frac{1}{2\omega}\frac{\partial}{\partial\omega} W_{\text{phys.}}[\omega].$$

Q4 Photon Propagator.

Consider the pure gauge Lagrangian with a gauge-fixing term:

$$\mathcal{L}(x) = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2.$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

a) Derive the equation of motion for the photon field A_μ.

b) Show that the appropriate photon propagator is

$$D_{\mu\nu}(k) = \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right).$$