QFT Spring2025

HW2

Q1 Fermi Golden Rule

Consider a quantum system with an unperturbed Hamiltonian H_0 and a weak interaction that is **adiabatically switched on**:

$$H = H_0 + V e^{\epsilon t}, \quad \text{with } \epsilon \to 0^+.$$

Initially, the system is in the eigenstate $|i\rangle$ of H_0 with energy E_i.

- Compute the survival amplitude

  $$c_i(t) = \langle i|U(t,0)|i\rangle$$

  up to second order in V, and show that it **diverges** as $\epsilon \to 0$.

- Show that the construction

  $$\frac{d}{dt} \ln c_i(t)$$

  is **finite**, demonstrating that the divergence cancels.

- This motivates the definition of the **renormalized self-energy**

  $$\Sigma_i = \frac{i}{t} \ln c_i(t).$$

  Show that the **energy shift** and **decay rate** can then be identified as

  $$\Delta E_i = \mathrm{Re}\,\Sigma_i, \quad \Gamma_i = -2\,\mathrm{Im}\,\Sigma_i.$$

- Derive the explicit expressions for ΔE_i and Γ_i:

  $$\Delta E_i = V_{ii} + \sum_{m \neq i} \mathcal{P} \frac{|V_{mi}|^2}{E_i - E_m},$$
  $$\Gamma_i = \frac{2\pi}{\hbar} \sum_{m \neq i} |V_{mi}|^2 \delta(E_m - E_i).$$

- To leading order in Σ_i show that

  $$|c_i(t)|^2 + \Gamma_i t = 1.$$

Q2 Schwinger proper time regularization.

- Show that

$$\int_{-\infty}^{\infty} dx\, \frac{1}{x^2+a^2} = (2\pi)\,\frac{1}{2a}.$$

- Make sense of the following identities:

$$\mathcal{A}^{-1} = \int_0^\infty dt\, e^{-t\mathcal{A}}$$
$$\ln \mathcal{A} = -\int_0^\infty dt\, \frac{1}{t}\left(e^{-t\mathcal{A}} - e^{-t\mathcal{I}}\right).$$

- With $\mathcal{A} \to p_4^2 + \omega^2$, show (again!) that

$$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi}\, \frac{1}{p_4^2+\omega^2} = \frac{1}{2\sqrt{\omega^2}}.$$

Q3 Self-energy (part I).

- Explain why

$$\mathrm{Im}\left(\ln(-1 \pm i\delta)\right) \to \pm\pi$$

  where δ is a small (positive) number. Select any numerical method and verify these results.

- A QFT computation of the self energy Σ_R of a resonance (decaying to two daughters of equal mass m) gives

$$\Sigma_R(s) = 2 \times \frac{g^2}{16\pi^2} \int_0^1 dx\, \ln\left(m^2 - x(1-x)s - i\delta\right).$$

  (The factor of 2 is the symmetry factor for identical particles.) Show that the self energy develops an imaginary part when the COM energy is above the threshold:

$$\mathrm{Im}\Sigma_R(s) = -2 \times \frac{1}{2}\, g^2\, \phi_2(s)\, \theta(\sqrt{s} - 2m)$$

  where $\phi_2(s)$ is the 2-body phase space. (HW01 Q4).

Q4 Recursive formula for N-body phase spaces.

The N-body Lorentz Invariant phase space (LISP) is defined as

$$\phi_N(s=P^2) = \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}\cdots\frac{d^3p_N}{(2\pi)^3}\frac{1}{2E_N}\times$$
$$(2\pi)^4\,\delta^4(P-\sum_i p_i).$$

Note that $E_j = \sqrt{p_j^2+m_j^2}$.

- Show that the 3-body phase space satisfies the recursion formula

$$\phi_3(s,m_1^2,m_2^2,m_3^2) = \int_{s'_-}^{s'_+}\frac{ds'}{2\pi}\,\phi_2(s,s',m_3^2)\;\phi_2(s',m_1^2,m_2^2).$$

Work out the limits: $s'_\pm$.

Recall the 2-body phase space $\phi_2(s,m_1^2,m_2^2)$ is given by

$$\phi_2(s,m_1^2,m_2^2) = \frac{q(s)}{4\pi\sqrt{s}}$$
$$q(s) = \frac{1}{2}\sqrt{s}\sqrt{1-\frac{(m_1+m_2)^2}{s}}\sqrt{1-\frac{(m_1-m_2)^2}{s}}.$$

- Derive (again) the result for the massless case:

$$\phi_3(s) = \frac{s}{256\pi^3}.$$

Q5 Fourier transform.

a) Show that

$$\begin{aligned} G(\vec{x}',\vec{x}) &= \langle\vec{x}'|\frac{1}{E-\frac{-\nabla^2}{2m_R}\pm i\delta}|\vec{x}\rangle \\ &= -2m_R\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2-q^2\mp i\delta}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \\ &= -2m_R\,\frac{1}{4\pi R}\,e^{\pm iqR}. \end{aligned}$$

where $R=|\vec{x}-\vec{x}'|$ and $E=\frac{q^2}{2m_R}$.

b) Show that $\frac{e^{-mr}}{r}$ and $\frac{4\pi}{k^2+m^2}$ is a pair of (3D) Fourier transform.