SU(2) chiral symmetry

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$SU_V(2) \leftrightarrow$ isospin symmetry

$$q \to e^{-i\alpha_V^a \tau^a} q \qquad \left(\tau^a = \frac{1}{2}\sigma^a\right)$$

q ↳ 4-spinor

$\to \begin{pmatrix} u \\ d \end{pmatrix}$ structure

$$\mathcal{L}_0 = \bar{q}(i\partial\!\!\!/ - \hat{m})q$$

$$\hat{m} = \begin{bmatrix} m_u & \\ & m_d \end{bmatrix}$$

$$\mathcal{H}_0 = -i\bar{q}\vec{\gamma}\cdot\nabla q + \bar{q}\hat{m}q$$

$$\Delta\mathcal{L}_0 \to \bar{q}\, e^{i\alpha_V^a\tau^a}(i\partial\!\!\!/ - \hat{m})\, e^{-i\alpha_V^a\tau^a} q$$

$$- \bar{q}(i\partial\!\!\!/ - m)q$$

$$\hat{=} -i\alpha_V^a\, \bar{q}\,[\tau^a, \hat{m}]\, q$$

$$\Rightarrow \Delta\mathcal{L}_0^a = \Big[\, i\tfrac{1}{2}(m_u - m_d)(\bar{u}d - \bar{d}u),$$

$$\tfrac{1}{2}(m_u - m_d)(\bar{u}d - \bar{d}u),$$

$$0 \,\Big]$$

vector structure

note that

$$\Delta \mathcal{H}^a = -\Delta \mathcal{L}^a$$

& $\Delta \mathcal{L}^a \neq 0$ if $m_u \neq m_d$

& Noether current does <u>not</u> conserve.

$$j_V^{\mu a} = \bar{q} \gamma^\mu \tau^a q$$

Lorentz (μ) & isospin (a) ~~structures~~

&

$$\partial_\mu j_V^{\mu a} \to \partial_t j_V^{a0} = \Delta \mathcal{L}^a \quad (\neq 0)$$

$$Q_V^a = \int d^3x \, j_V^{a0}$$

$$= \int d^3x \, \bar{q}_x \gamma^0 \tau^a q_x$$

↳ can take $t=0$

& it follows that

$$[Q_V^a, \overset{\int_{\vec{x}} \mathcal{H}}{H}] = i\partial_t Q^a = \int_{\vec{x}} i \Delta \mathcal{L}^a$$

$$= \int_{\vec{x}} -i \Delta \mathcal{H}^a$$

$$= \int_{\vec{x}} \bar{q} [\tau^a, \hat{M}] q$$

$SU_A(2) \longleftrightarrow$ chiral symmetry

$$q \rightarrow e^{-i\gamma_5 \alpha^a_{AV} \tau^a} q$$

$$\gamma_5^+ = \gamma_5$$

$$\Delta\mathcal{L}_0 \simeq i\alpha^a_{AV} \, \bar{q}\gamma_5 \{\tau^a, \hat{m}\} q$$

$$\partial_\mu j^{\mu a}_{AV} = \Delta\mathcal{L}^a.$$

$$Q^a_{AV} = \int d^3x \, \bar{q}\gamma^0\gamma_5\tau^a q.$$

$$[Q^a_{AV}, H] = i\partial_t Q^a_{AV} = \int_V \bar{q}\gamma_5\{\tau^a, \hat{m}\} q$$

$\rightarrow 0$ only if

$\hat{m} \rightarrow 0$

massless quarks.

$\rightarrow$ chiral limit

we can verify all these directly

$$Q^a_{AV} = \int d^3x \; \bar{q}_x \gamma^0 \gamma_5 \tau^a q_x$$

$$H = \int d^3x \; \mathcal{H}$$

using $\{ q^a_{\vec{x}}, q^{b\dagger}_{\vec{y}} \} = \delta^3_{\vec{x}-\vec{y}} \, \delta^{ab}$ equal time

0 for other combinations

$$[Q^a_{AV}, H]$$

other commute w Q^a_{OV}

$$= \int\int [\bar{q}_x \gamma^0 \gamma_5 \tau^a q_x, \; \bar{q}_y \hat{m} q + \cdots]$$

$$= \int\int \bar{q} \gamma^0 \gamma_5 \tau^a q \;\; \bar{q} \hat{m} q$$

$$= \int \bar{q} (\gamma^0 \gamma_5 \tau^a \gamma^0 \hat{m} - \hat{m} \gamma_5 \tau^a) q$$

$$= -\int \bar{q} \gamma_5 \{ \tau^a, \hat{m} \} q$$

$$= -i \; \Delta H^a$$

PML trick

$$i\partial_t Q^a = [Q^a, H]$$

H is generator of time evolution

$$i\partial_{\alpha_a} H = [H, Q^a]$$

how H Δ under the symmetry operation

Q^a is generator

$$[Q^a, H] = -i\,\Delta H^a = i\,\Delta L^a$$

obtained by running the chiral rotation

Spontaneous Symmetry Breaking

$$\mathcal{U} = e^{-i\alpha^a Q^a}$$

if $A \to 0$ $\quad [Q^a, H] = 0$

$$|0'\rangle = \mathcal{U}|0\rangle$$

$$\langle 0'|H|0'\rangle = \langle 0|\mathcal{U}^\dagger H \mathcal{U}|0\rangle$$

$$= \langle 0|H|0\rangle$$

$E'_{GS} = E_{GS}$

Normally $|0'\rangle = |0\rangle$ or $Q|0'\rangle = 0$

SSB $\quad Q|0'\rangle \neq 0$

this state is special

$$H \underline{Q|0'\rangle} = E_{GS} \underbrace{Q|0'\rangle}_{\neq 0}$$

$Q|0'\rangle$ has same energy as in vac.

$\longrightarrow$ X cost energy to excite

$\Rightarrow$ Goldstone's Boson

An order parameter $\hat{O}$ can be an operator that

$$[\hat{O}, Q] \neq 0$$

$$\langle 0' | \hat{O} | 0' \rangle \neq \langle 0 | \hat{O} | 0 \rangle$$

$\longrightarrow$ can be used to probe vac. ~~structure~~

e.g.

$$q\bar{q} \to q\bar{q} - \alpha^a_{AV} \left(2i \bar{q} \gamma_5 \tau^a q \right)$$

$$[q\bar{q}, Q^a_{AV}] = i \frac{\partial}{\partial \alpha^a_{AV}} q\bar{q}$$
$$= 2 \bar{q} \gamma_5 \tau^a q$$

$\langle 0' | q\bar{q} | 0' \rangle$ is one choice of order parameter

SSB :

$[Q, H] = 0$ symmetry exists.

but

$Q | 0' \rangle \neq 0$ Not respected by vac.

consequences of SSB :

→ massless Goldstone bosons

→ Gellmann - Oakes - Renner

→ PCAC $m_q \approx 0$

$$\bar{q}q \xrightarrow{AV} \bar{q}q - \alpha^a\, 2i\, \bar{q}\gamma_5 \tau^a q$$

$$2i\, \bar{q}\gamma_5 \tau^a q \xrightarrow{AV} 2i\, \bar{q}\gamma_5\tau^a q + \alpha^a\, \bar{q}q$$

this acts like a rotation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\simeq \begin{pmatrix} x - \theta y \\ y + \theta x \end{pmatrix}$$

$\Rightarrow$ $(\bar{q}q)^2 + (2i\, \bar{q}\gamma_5 \tau^a q)^2$ is invariant

if we introduce (σ, π^a) such that

$$\left.\begin{array}{l} \sigma \xrightarrow{AV} \sigma + \alpha^a_{AV}\, \pi^a \\ \pi^a \longrightarrow \pi^a - \alpha^a_{AV}\, \sigma \end{array}\right\}$$ rotate oppositely

s.t.

$\bar{q}\left(\sigma - 2i\, \gamma_5 \tau^a \pi^a\right) q$ is invariant

$$\Rightarrow \bar{q}'\left(\sigma' - 2i\gamma_5\tau^a\pi^{a\prime}\right) q' = \bar{q}\left(\sigma - 2i\gamma_5\tau^a\pi^a\right) q$$

also

$\sigma^2 + \pi^{a2}$; $(\bar{q}q)^2 + (2i\bar{q}\gamma_5\tau^a q)^2$

are invariant!

Summary of SU(2) rotations

$$\vec{\alpha}_V \times 2i\,\bar{q}\gamma_5\vec{\tau}q$$

$$\bar{q}q \xrightarrow{V} \bar{q}q$$

$$2i\,\bar{q}\gamma_5\tau^a q \xrightarrow{V} 2i\,\bar{q}\gamma_5\tau^a q + \varepsilon^{abc}\alpha_V^b\, 2i\,\bar{q}\gamma_5\tau^c q$$

$$\bar{q}q \xrightarrow{AV} \bar{q}q - \alpha_{AV}^a\,(2i\,\bar{q}\gamma_5\tau^a q)$$

$$2i\,\bar{q}\gamma_5\tau^a q \xrightarrow{AV} 2i\,\bar{q}\gamma_5\tau^a q + \alpha_{AV}^a\,\bar{q}q$$

$$\sigma \xrightarrow{V} \sigma$$

$$\vec{\pi} \xrightarrow{V} \vec{\pi} + \vec{\alpha}_V \times \vec{\pi}$$

$$\sigma \xrightarrow{AV} \sigma + \vec{\alpha}_V \cdot \vec{\pi}$$

$$\vec{\pi} \xrightarrow{AV} \vec{\pi} - \vec{\alpha}_V\,\sigma$$

$$\bar{q}\,2i\gamma_5\vec{\tau}\,q$$

$$-\,\vec{\alpha}_V \times \vec{\pi} \cdot (\vec{\pi}_q)$$

$$\bar{q}\,(\sigma - \vec{\pi}\cdot 2i\gamma_5\vec{\tau})\,q$$

$$\left.\begin{aligned} \xrightarrow{V} \bar{q}\,(\sigma - \vec{\pi}\cdot 2i\gamma_5\vec{\tau})\,q \;&-\; \Delta\vec{\pi}\cdot \bar{q}\,2i\gamma_5\vec{\tau}\,q \\ &-\; \vec{\pi}\cdot\Delta\,(\bar{q}\,2i\gamma_5\vec{\tau}\,q) \end{aligned}\right\}\ \text{they cancel}$$

$$-\,\vec{\pi}\cdot(\vec{\alpha}_V \times \vec{\pi}_q)$$

Current Algebra

$$Q_V^a = \int d^3x \; \bar{q}\gamma^0 \tau^a q$$

$$Q_{AV}^a = \int d^3x \; \bar{q}\gamma^0\gamma_5 \tau^a q$$

$$[Q_V^a, Q_V^b] = i\varepsilon^{abc} Q_V^c$$

$$[Q_{AV}^a, Q_{AV}^b] = i\varepsilon^{abc} Q_{AV}^c$$

$$[Q_V^a, Q_{AV}^b] = i\varepsilon^{abc} Q_{AV}^c$$

if we define

$$Q_{\substack{L\\R}} = \frac{1}{2}(Q_V \pm Q_{AV})$$

$$[Q_L^a, Q_L^b] = i\varepsilon^{abc} Q_L^c$$

$$[Q_R^a, Q_R^b] = i\varepsilon^{abc} Q_R^c$$

$$[Q_L^a, Q_R^b] = 0$$

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L, R decoupled

$\longrightarrow$ L, R : good language to understanding the symmetry

$$q_{R}^{L} = \underbrace{\frac{1}{2}(I \pm \gamma_5)}_{\hat{P}_{L/R}\ \text{projector}} q$$

$$\overline{q_L} = (\hat{P}_L q)^\dagger \gamma^0$$

$$= \overline{q}\,\hat{P}_R$$

$$\hat{P}_L \hat{P}_R = \hat{P}_R \hat{P}_L = 0$$

$$\overline{q} q = \overline{q_L} q_R + \overline{q_R} q_L \longrightarrow \text{they mix !}$$

$$\overline{q}\gamma^\mu q = \overline{q_L}\gamma^\mu q_L + \overline{q_R}\gamma^\mu q_R \longrightarrow \text{they don't mix !}$$

$L, R \longleftrightarrow V, AV$ via

$$U_L \longleftrightarrow e^{-i(\vec{\alpha}_V + \vec{\alpha}_{AV})\cdot\vec{\tau}}$$

$$U_R \longleftrightarrow e^{-i(\vec{\alpha}_V - \vec{\alpha}_{AV})\cdot\vec{\tau}}$$

for V

$$U_L \longrightarrow e^{-i\vec{\alpha}_V\cdot\vec{\tau}}$$

$$U_R \longrightarrow e^{-i\vec{\alpha}_V\cdot\vec{\tau}}$$

for AV

$$U_L \longrightarrow e^{-i\vec{\alpha}_{AV}\cdot\vec{\tau}}$$

$$U_R \longrightarrow e^{i\vec{\alpha}_{AV}\cdot\vec{\tau}}$$

$$= U_L^\dagger$$

$$\Sigma = \sigma \mathbb{I}_2 + \vec{\pi} \cdot 2i \vec{\tau}$$

$$P_{\substack{L\\R}} = \frac{1}{2}(\mathbb{I} \pm \gamma_5)$$

$$\Sigma \to U_L \Sigma U_R^\dagger \qquad \text{s.t.}$$

$$\bar{q}_L \Sigma q_R + \bar{q}_R \Sigma^\dagger q_L \qquad \text{is invariant by construction.}$$

now :

$$\to \bar{q}_L (\sigma + 2i \vec{\pi}\cdot\vec{\tau}) q_R + \bar{q}_R (\sigma - 2i \vec{\pi}\cdot\vec{\tau}) q_L$$

$$= \sigma \bar{q} q + \bar{q}(-\vec{\pi}\cdot 2i\vec{\tau}) \underbrace{(\hat{P}_L - \hat{P}_R)}_{\gamma_5} q$$

$$= \bar{q} (\sigma - 2i \vec{\tau}\gamma_5 \cdot \vec{\pi}) q$$

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how $\sigma, \vec{\pi}$ transform ?

$$\Sigma \xrightarrow{V} U_V \Sigma U_V^\dagger$$

$$= U_V (\sigma \mathbb{I}_2 + \vec{\pi} \cdot 2i \vec{\tau}) U_V^\dagger$$

$$= \underset{\uparrow \sigma'}{\sigma \mathbb{I}_2} + i \underset{\uparrow \vec{\pi}'}{(\vec{\pi} + \vec{\alpha}_V \times \vec{\pi})} \cdot 2\vec{\tau}$$

not $U_{AV}^\dagger$

$$\Sigma \xrightarrow{AV} U_{AV}\,\Sigma\, U_{AV}$$

$$= U_{AV}\,(\sigma + \vec{\pi}\cdot 2i\vec{\tau}\,)\, U_{AV}$$

$$\approx \sigma - 2i\,\vec{\alpha}_{AV}\cdot\vec{\tau}\,\sigma +$$

$$i\,\vec{\pi}\cdot 2\vec{\tau} + 2\alpha^b_{AV}\,\pi^a\,\{\tau^b, \tau^a\}$$

$$= (\sigma + \vec{\alpha}_{AV}\cdot\vec{\pi}\,) +$$

$$(\vec{\pi} - \sigma\vec{\alpha}_{AV}\,)\cdot 2i\vec{\tau}$$

$$\sigma \to \sigma + \vec{\alpha}_{AV}\cdot\vec{\pi}$$

$$\vec{\pi} \to \vec{\pi} - \vec{\alpha}_{AV}\,\sigma$$

even more symmetry start:

$$\sigma = \frac{\mathrm{tr}\,\Sigma}{\mathrm{tr}\,\mathbb{I}_2}$$

$$\Sigma \to U_L\,\Sigma\, U_R^\dagger$$

easy $\Sigma \xrightarrow{V} U_V\,\Sigma\, U_V^\dagger$

$\mathrm{tr}\,\Sigma \to \mathrm{tr}\,\Sigma$ $\quad$ σ is invariant

$$\Sigma \xrightarrow{AV} U_{AV}\,\Sigma\, U_{AV}$$

$\mathrm{tr}\,\Sigma \to$ not <u>invariant</u> at all

G. O. R.

$$M_\pi^2 = \frac{-1}{f_\pi^2} \langle 0 | [Q_{AV}^a, \overset{\text{no sum}}{[Q_{AV}^a, H]}] | 0 \rangle$$

$$[Q_{AV}^a, [Q_{AV}^a, H]] = \int d^3x\, \bar{q}_x \{\tau^a, \{\tau^a, \hat{m}\}\} q_x$$

— G. O. R

$$\underbrace{M_\pi^2}_{140\text{ MeV}} \approx \frac{1}{\underbrace{f_\pi^2}_{93\text{ MeV}}} \Big(\underbrace{m_u}_{\text{few MeV}} \underbrace{\langle \bar{u}u \rangle}_{-(250\text{ MeV})^3 \text{ each}} + m_d \underbrace{\langle \bar{d}d \rangle}_{} \Big)$$

Amazing lever ~~of~~ scale

$\langle \bar{q}q \rangle$ reveals an Energy scale
→ how protons get the mass!

$$\{\tau^a, \{\tau^a, \hat{m}\}\} = \begin{bmatrix} \frac{m_u+m_d}{2} & \\ & \frac{m_u+m_d}{2} \end{bmatrix} \text{ if } a = x, y$$

$$= \begin{bmatrix} m_u & \\ & m_d \end{bmatrix} \text{ if } a = z$$

Goldberger – Treiman Relation

Ref bk
P. 329.

$$\mathcal{L} = \bar{q}(i\partial\!\!\!/ - \hat{m})q - g(\bar{q}q\,\sigma - \bar{q}\,2i\gamma_5\vec{\tau}q\cdot\vec{\pi})$$

$$- V(\sigma^2 + \vec{\pi}^2)$$

$\updownarrow$ Nucleons

$$\mathcal{L}_N = \bar{N}(i\partial\!\!\!/ - m_N)N - g_{NN\sigma}\bar{N}N\sigma +$$

$$\underbrace{g_A\,\bar{N}\gamma^\mu\gamma_5\tau^a N}_{\Sigma^2}\ \underbrace{A^a_{\pi\,\mu}}_{\Sigma^3}\ \frac{1}{f_\pi^2} + \cdots$$

$\swarrow$ $\qquad\qquad\qquad \searrow (-f_\pi\,\partial_\mu\vec{\pi})$

$$\mathcal{L}_N^{int} = \frac{g_A}{f_\pi}\left[\partial_\mu(\bar{N}\gamma^\mu\gamma_5\tau^a N)\right]\pi^a$$

$$(i\partial\!\!\!/ - m_N)N = 0$$

$$\omega \qquad \bar{N}(i\overleftarrow{\partial\!\!\!/} + m_N) = 0$$

$$\longrightarrow\ i\,\frac{g_A}{f_\pi}\,\bar{N}\gamma_5\{\hat{m}_N, \tau^a\}N\,\pi^a$$

$$\simeq \frac{g_A}{f_\pi}m_N\ \bar{N}\,2i\gamma_5\tau^a N\ \pi^a$$

this is the form we expect

$$g_{NN\pi} \to \frac{g_A}{f_\pi} m_N \qquad g_A \sim 1.26$$

for beta decay

if we take

$$A^a_\mu = g_A \ \bar{N} \gamma_\mu \gamma_5 \tau^a N - f_\pi \partial_\mu \pi^a$$

$$\partial^\mu A^a_\mu \simeq m_N g_A \ \bar{N}\, 2i\gamma_5 \tau^a N - f_\pi \partial^2 \pi^a$$

$$\simeq f_\pi m_\pi^2 \pi^a$$

ext. current

$$(\partial^2 + m_\pi^2)\pi^a \simeq \frac{m_N g_A}{f_\pi} \bar{N}\, 2i\gamma_5 \tau^a N$$

$$\frac{g^2_{\pi NN}}{4\pi} \sim 13.8$$

13.169

$$\frac{g_A m_N}{f_\pi} = g_{\pi NN} \longleftrightarrow g_A = 1.31$$

$$\mathcal{L}_{\pi} \rightarrow \frac{1}{2}(\partial\vec{\pi})^2 - \frac{1}{2}m_\pi^2\vec{\pi}^2 + g\,\vec{\pi}\cdot\vec{J}_N$$

E.O.M. $$(\partial^2 + m_\pi^2)\vec{\pi} = g\,\vec{J}_N$$

$$\partial_\mu A^\mu_a = f_\pi m_\pi^2 \pi^a.$$

$$A^\mu_a = -\underline{f_\pi}\,\partial^\mu \pi^a$$

$$\leftrightarrow \langle\sigma\rangle$$

$$\pi^a \rightarrow \pi^a - f_\pi \alpha^a_{AV}.$$

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Notes on $U_A(1)$ anomaly

Ref. ①

a) red bk v. 2

b) Ynduráin.

$$q \to e^{-i\gamma_5 \alpha_A} q$$

$$-\bar{q}\hat{m}q \to -\bar{q}\hat{m}e^{-2i\gamma_5\alpha_A} q$$

$$\hat{=} -\bar{q}\hat{m}q + 2i\alpha_A \bar{q}\hat{m}\gamma_5 q$$

$$\Delta\mathcal{L} \ (\text{per } \alpha_A) = \underline{2i\,\bar{q}\hat{m}\gamma_5 q}.$$

$$\hookrightarrow m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d + m_s \bar{s}\gamma_5 s$$

$$j_5^\mu = \bar{q}\gamma^\mu\gamma_5 q$$

The expectation that

$$\partial_\mu j_5^\mu = \Delta\mathcal{L} = 2i\,\bar{q}\hat{m}\gamma_5 q$$

$\to 0$ in the chiral limit

is NOT realized due to the existence of Anomaly : Jacobian $\to$ eff. term in the Lagrangian

$$\partial_\mu j_5^\mu = \overbrace{i\,2\,\bar{q}\,\hat{m}\,\gamma_5\,q}^{\Delta\mathcal{L}_m} + \overbrace{\frac{N_f\,g^2}{16\pi^2}\,\tilde{G}\,G}^{\text{anomaly}}$$

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,G_a^{\rho\sigma}$$

(at least) 2 consequences

① η' fails to become a massless Goldstone bosons even in the chiral limit

$$m_{\eta'}^2 \to \frac{2N_f}{f_\eta^2}\,\chi^A_{YM}$$

② $\pi^0 \to \gamma\gamma$ decay

$$q_A = \frac{g^2}{32\pi^2}\,G\tilde{G}$$

$$\chi^A = \int d^4x\,\langle q_A(x)\,q_A(0)\rangle_c$$

$$\bar{q}\,\gamma^\mu\gamma_5\,\vec{\tau}\,q$$

$\to SU(2)$; look at τ^3

$$j^{\tau_3\,\mu} \to \bar{u}\gamma^0\gamma_5 u - \bar{d}\gamma^0\gamma_5 d$$

$$\partial_\mu j^{\tau_3\,\mu} \to 2i\,(m_u\,\bar{u}\gamma_5 u - m_d\,\bar{d}\gamma_5 d)$$

$$+\ \frac{\alpha\,N_c}{6\pi}\,F_{\mu\nu}\tilde{F}^{\mu\nu}$$

QED

testing N_c

for photon

θ-vacuum

$G\tilde{G} \longrightarrow \partial_\mu K^\mu$ total derivative $\leftrightarrow$. a surface term. in the Lagrangian

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left(A_\nu^a G_{\lambda\sigma}^a + \frac{1}{3} g f_{abc} A_\nu^a A_\lambda^b A_\sigma^c\right)$$

and the winding #

$$\frac{g^2}{32\pi^2}\int d^4x\, G\tilde{G}$$

$$\longrightarrow \frac{g^2}{32\pi^2}\int d^4x\, \partial_\mu K^\mu$$

$$\longrightarrow \frac{g^2}{32\pi^2}\int d^3x\, K^0 \Big|_{t=-\infty}^{t=+\infty}$$

$$= n_Q(+\infty) - n_Q(-\infty)$$

$\uparrow$ CS #

$\longrightarrow$ use n to label a gluon config. (vacuum).

$|n\rangle$

& a gauge transform will move around various n's.

the vac. $\longrightarrow$ need to be a combination of $|n\rangle$'s

$$|\theta\rangle = \sum_n e^{-in\theta} |n\rangle \rightarrow$$ origin of θ-vac

to specify QCD

$\longrightarrow$ $\mathcal{L}_{QCD} \oplus \Lambda_{QCD} \oplus \theta$

$\sim$ trace anomaly $\quad\quad \sim U_A(1)$ anomaly

how θ shows up?

recall $|\theta\rangle = \sum_n e^{-in\theta} |n\rangle$

$$\langle \theta=0 | \hat{O} | \theta=0 \rangle$$

$$= \sum_{m,n} \langle m | \hat{O} | n \rangle$$

$$\langle \theta | \hat{O} | \theta \rangle$$

$$= \sum_{m,n} e^{i\underline{(m-n)}\theta} \langle m | \hat{O} | n \rangle$$

$$\frac{g^2}{32\pi^2} \int d^4x \, G\tilde{G}$$

$$\Rightarrow \mathcal{L}_\theta = \mathcal{L}_{\theta=0} + \theta \frac{g^2}{32\pi^2} \int d^4x \, G\tilde{G}$$

a surface term.

$\underline{P\text{odd} \quad T\text{odd}}$

Under $U_A(1)$

$$\Delta \mathcal{L}_m = 2i\alpha_A \bar{q} \hat{M} \gamma_5 q$$

$$\partial_\mu j_5^\mu = \Delta \mathcal{L}_m + N_f \frac{g^2}{16\pi^2} G\tilde{G}$$

in the chiral limit $\quad \Delta \mathcal{L}_m \to 0$

$$\int d^4x \, \partial_\mu j_5^\mu = Q_5(t=+\infty) - Q_5(t=-\infty)$$

$$\Delta Q_5 \simeq 2N_f \, \Delta n$$

Now for a gauge transform $\mathcal{U}_1$

$\mathcal{U}_1 |n\rangle = |n+1\rangle$

$\rightarrow$

$$\mathcal{U}_1 |\theta\rangle = \sum_n e^{-in\theta} |n+1\rangle$$

$$= e^{i\theta} |\theta\rangle$$

Now consider a $\mathcal{U}_A(1)$ transformed $|\theta\rangle$

$$\mathcal{U}_1 \left(e^{-i\alpha Q_5} |\theta\rangle \right)$$

$$= \mathcal{U}_1 e^{-i\alpha Q_5} \mathcal{U}_1^{-1} \left(\mathcal{U}_1 |\theta\rangle \right)$$

we know $\mathcal{U}_1 |n\rangle \rightarrow |n+1\rangle$

$$Q_5 \xrightarrow{\mathcal{U}_1} Q_5 + 2N_f$$

or

$$\mathcal{U}_1^{-1} e^{-i\alpha Q_5} \mathcal{U}_1 = e^{-i\alpha (Q_5 + 2N_f)}$$

$$\mathcal{U}_1 \left(e^{-i\alpha Q_5} |\theta\rangle \right) = e^{-i\alpha (Q_5 + 2N_f)} e^{i\theta} |\theta\rangle$$

rhs $\rightarrow$ rearranged as

$$e^{i(\theta - 2N_f \alpha)} \left(e^{-i\alpha Q_5} |\theta\rangle \right)$$

$$\Rightarrow \quad e^{-i\alpha Q_5} |\theta\rangle = |\theta - 2N_f \alpha\rangle$$

the action of a $U_A(1)$ rotation

$\updownarrow$

shift of $\theta \rightarrow \theta - 2N_f \alpha$.

if $m \rightarrow 0$

α cannot be physical

$\rightarrow$ θ cannot be physical.

if $m \not\rightarrow 0$

hell

$$\bar{\theta} = \theta + \arg \det \mathcal{M}$$

...ery small
...n our
...orld

gluonic

quark

they have to cancel.

$\rightarrow$ ...

$G\tilde{G} \sim \vec{E}\cdot\vec{B}$

C, P odd

weak int has the bad habit of CP odd

→ brought it to QCD ?

χ^A_{YM} → measured on the lattice

$M^2_{\eta'} = \frac{2N_f}{f^2_\eta}\chi^A_{YM}$ is verified too

$$\Rightarrow \quad \chi^A_{YM} \approx \frac{\partial^2 E(\theta)}{\partial\theta\,\partial\theta}$$

or

$$E(\theta) = E_{\theta=0} + \frac{1}{2}\theta^2\chi^A_{YM}$$

how θ is "unphysical"

when χ^A_{YM} is very real ?

strong CP problem ...