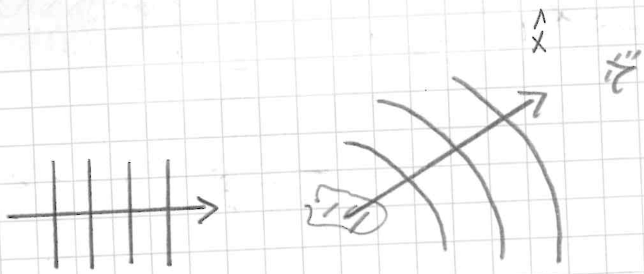
Scattering Theory



$$E = \frac{q^2}{2m_R}$$

$$e^{i\vec{q}\cdot\vec{r}} \qquad \frac{e^{iqr}}{r} f$$

$$\psi_{\vec{x}} \xrightarrow{r\to\infty} e^{i\vec{q}_i\cdot\vec{x}} + \frac{e^{iqr}}{r} f_{\vec{q}_f}$$

$$\vec{q}_f = q\,\hat{x}$$

$$H|\psi\rangle = (H_0 + V)|\psi\rangle = E|\psi\rangle$$

$$(E - H_0)|\psi\rangle = V|\psi\rangle$$

$$\Rightarrow |\psi\rangle = |\varphi\rangle + \frac{1}{E - H_0} V|\psi\rangle$$

satisfies

$$(E - H_0)|\varphi\rangle = 0$$

$$\langle\vec{x}|\psi\rangle = \langle\vec{x}|\varphi\rangle + \langle\vec{x}|\frac{1}{E-H_0} V|\psi\rangle$$

$$e^{i\vec{q}\cdot\vec{x}} \qquad \int d\vec{x}'\, \langle\vec{x}|\frac{1}{E-H_0}|\vec{x}'\rangle\, V(\vec{x}')\, \psi(\vec{x}')$$

$$r \to \infty$$

$$\Leftrightarrow \frac{1}{r} e^{iqr} f_q$$

$$G^0_{\vec{x}\vec{x}'} = \langle \vec{x} | \frac{1}{E - H_0 \underline{+ i\delta}} | \vec{x}' \rangle$$

$\hookrightarrow$ to properly define the Green's function

$$H_0 = \frac{-\nabla^2}{2M_R}$$

$$G^0_{\vec{x}\vec{x}'} = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{E - \frac{\vec{p}'^2}{2m_R} + i\delta} e^{i\vec{p}'\cdot(\vec{x}-\vec{x}')}$$

$$= 2M_R \frac{-1}{4\pi|\vec{x}-\vec{x}'|} e^{iq|\vec{x}-\vec{x}'|}$$

$$//$$

$$\psi_{\vec{x}} = \varphi_{\vec{x}} + \int d^3x' \frac{-2m_R}{4\pi} \frac{e^{iq|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} V_{\vec{x}'} \psi_{\vec{x}'}$$

$r \to \infty$

$$\longrightarrow \varphi_{\vec{x}} + \frac{e^{iqr}}{r} \underline{\int d^3x' \frac{-2m_R}{4\pi} e^{-iq\hat{x}\cdot\vec{x}'} V_{\vec{x}'} \psi_{\vec{x}'}}$$

f_q is directly identified

Born Approx

$\psi_{\vec{x}'}$ in the integral $\longrightarrow$ $\psi_{\vec{x}'} = e^{i\vec{q}_i \cdot \vec{x}'}$

$$f_q^{Born} = \frac{-2m_R}{4\pi} \int d^3x' \; e^{-i\Delta\vec{q}\cdot\vec{x}'} \, V_{\vec{x}'}$$

$$\longrightarrow \frac{-2m_R}{2\pi} \tilde{V}_{\Delta\vec{q} = \vec{q}_f - \vec{q}_i}$$

$f_q \leftrightarrow$ - of the Fourier transform of the potential

$$f_q = \frac{-2m_R}{4\pi} \int d^3x' \; e^{-iq\hat{x}\cdot\vec{x}'} \, V_{\vec{x}'} \, \psi_{\vec{x}'}$$

remains a general way to compute f_q

Another View : T-matrix

$$|\psi\rangle = |\varphi\rangle + \frac{1}{E-H_0} V|\psi\rangle$$

$$V|\psi\rangle = V|\varphi\rangle + V\frac{1}{E-H_0}V|\psi\rangle$$

$$= V|\varphi\rangle + V\frac{1}{E-H_0}V|\varphi\rangle +$$

$$V\frac{1}{E-H_0}V\frac{1}{E-H_0}V|\varphi\rangle + \cdots$$

$$= \{V + V\hat{G}_0 V + V\hat{G}_0 V \hat{G}_0 V$$

$$\updownarrow \quad + \cdots\} \; |\varphi\rangle$$

$$T := \frac{V}{1-G_0 V} = V + VG_0 T$$

$$G_0^{\pm} = \frac{1}{E-H_0 \pm i\delta}$$

$\Rightarrow$ $V|\psi\rangle = T|\varphi\rangle$

T works on std. basis $|\varphi\rangle$
& contain all info. in $|\psi\rangle$

$T = V + VG_0 T$ $\longrightarrow$ Lippmann - Schwinger eqn.

$\Rightarrow$ get rid of $|\psi\rangle, |\varphi\rangle$

this yields another way to compute f_q :

$$\psi_{\vec{x}} = \varphi_{\vec{x}} + \int d^3x' \, G^0_{\vec{x}\vec{x}'} \langle \vec{x}' | T | \vec{q}_i \rangle$$

$r \to \infty$

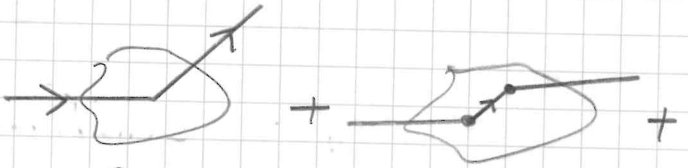
$$\to \varphi_{\vec{x}} + \frac{e^{iqr}}{r} \underbrace{\langle \vec{q}_f | T | \vec{q}_i \rangle \left(\frac{-2M_R}{4\pi} \right)}_{f_q}$$

& $T \to V$

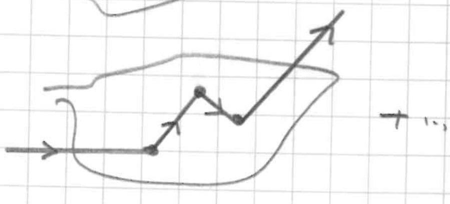
$\Longleftrightarrow$

Born approx.

T-matrix   T stands for transfer matrix



Born



higher order term.

$$|\psi\rangle = |\varphi\rangle + G_0 V |\psi\rangle$$
$$= (I + G_0 T) |\varphi\rangle$$

$|\psi\rangle \longleftrightarrow |\varphi\rangle$

translation

Yet another view : resolvent G

$$G = \frac{1}{E - H}$$

$$G^0 = \frac{1}{E - H_0}$$

$$G^{-1} = G_0^{-1} - V$$

$$G = \frac{1}{G_0^{-1} - V}$$

$$= G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots$$

$$= G_0 + G_0 T G_0$$

see $VG = TG_0$. ?

$$G = G_0 + G_0 T G_0$$

$\downarrow$

full Green's function

e.g.

$\rho_E = -2 \operatorname{Im} G$ to study spectral function

$\text{Im}\,T \neq 0$ even when $\text{Im}\,V = 0$

how $\text{Im}\,T$ is generated ?

T (or f) has to be complex to satisfy optical theorm

$$f = \frac{1}{q} e^{i\delta} \sin\delta$$

$$\Rightarrow \frac{1}{q} \text{Im} f = |f|^2$$

$\text{Im}\, G_0 \simeq 0$ s.t. $\text{Im}\, G \neq 0$

$\longrightarrow$ Width ?!

How ?

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots$$

resum the series to get a width

$$\text{Im}\, G_0^+ \longrightarrow -\pi\, \delta_{E-\hat{H}_0}$$

$$-2\, \text{tr}\, \text{Im}\, G_0^+ \rightarrow \text{tr}\, 2\pi\, \delta_{E-\hat{H}_0}$$

$\longleftrightarrow$ ρ_{NR} phase space

$$\rho_{NR} = \int \frac{d^3q}{(2\pi)^3} 2\pi\, \delta_{E-\frac{q^2}{2m_R}}$$

$$= \frac{m_R q}{\pi}$$

Summary

$H_{int} \to T$-matrix

$$f_q = \frac{-2m_e}{4\pi} \langle \vec{q}_f | T | \vec{q}_i \rangle$$

$$T = V + V G_0 T$$

$\angle S$ $q \to$

a natural $-$ sign between T & f

$$f_q \leftrightarrow \frac{1}{q} \sin\delta \, e^{i\delta}$$

or

$$S = e^{2i\delta} = 1 + 2iq f_q$$

$$= 1 + i \frac{2\pi \, \varepsilon_{NR}}{m_R} f_q$$

$$= 1 - i \, \varepsilon_{NR} T$$

it can be generalized: see scat. theory notes

$$\hat{S} = I - i \, 2\pi \, \delta_{E-H_0} \hat{T}$$

$$e^{2i\delta} \leftrightarrow 1 - i \varepsilon T \leftrightarrow 1 + 2iqf$$

$$\varepsilon \to \frac{q}{4\pi\sqrt{s}} \qquad T = -8\pi\sqrt{s} f$$

convention :

Scattering theorist need

connect to LS

$$\hat{S} = I - i\, 2\pi\, \delta_{E-H_0}\, \hat{T}_{scat}$$

$$= \underbrace{G_0^* G^{*-1}}_{\Omega_2^{-1}} \underbrace{G\, G_0^{-1}}_{\Omega_1}$$

$$G_0 = \frac{1}{E - H_0 + i\delta}$$

$$G = \frac{1}{E - H + i\delta}$$

$$\mathrm{tr} \ln \hat{S} = \ln \left[1 - i\, \mathcal{C}\, \langle \hat{T}_{scat} \rangle \right]$$

QFT people need ...

$$\hat{S} = I + i (2\pi)^4 \delta^4_{P_f P_i}\, M_{QFT}$$

$iM_{QFT} \leftrightarrow$ Feynman rules.

$M_{QFT} \leftrightarrow - \hat{T}_{scat}$

is unfortunate !!

self energy check.

$$\Sigma \sim -4\pi f \, n$$

$$f \simeq \frac{-1}{4\pi} 2m_R T$$

$$\Rightarrow \Sigma \sim 2m_R T \, n.$$

if $V < 0$ attractive int.

$f > 0$ $\quad f \simeq a_{\text{eff}} > 0$ $\quad$ ($q \to 0$)

$\Sigma < 0$

Good convention to follow

Appendix Quick way to $G^{\pm}_{\vec{x}}$:

$$G^{\pm} = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{E - \frac{p'^2}{2m_R} \pm i\delta} e^{i\vec{p}'\cdot\vec{x}}$$

$$= -2m_R \int \frac{d^3p'}{(2\pi)^3} \frac{1}{p'^2 - q^2 \mp i\delta}$$

using

$$\int \frac{d^3p'}{(2\pi)^3} \frac{1}{p'^2 + m^2} e^{i\vec{p}'\cdot\vec{x}} = \frac{1}{4\pi r} e^{-\sqrt{m^2}\, r}$$

$$G^{\pm} \to -2m_R \frac{1}{4\pi r} e^{-\sqrt{-q^2 \mp i\delta}\; r}$$

now

$$\sqrt{-q^2 \mp i\delta} = \mp i q$$

why?

e.g. $-q^2 - i\delta \Longrightarrow$

$-\pi$

$$\simeq q^2 e^{-i\pi}$$

$$\sqrt{} \longrightarrow q\, e^{-i\frac{\pi}{2}} = -iq$$

$$G^{\pm} = \frac{-2m_R}{4\pi} \frac{1}{r} e^{\pm iqr} \quad //$$

what if I don't want to use

FT $\frac{1}{p'^2+m^2} \leftrightarrow \frac{1}{4\pi r} e^{-mr}$

contour integral can be done directly

$$\int \frac{d^3p'}{(2\pi)^3} \frac{1}{p'^2 - q^2 \mp i\delta} e^{i\vec{p}'\cdot\vec{x}}$$

$$= \frac{2\pi}{8\pi^3}\int dp' \frac{1}{p'^2-q^2\mp i\delta}\, p'^2\, \frac{2\sin p'r}{p'r}$$

$$= \frac{1}{4\pi^2}\frac{1}{r}\int_{-\infty}^{+\infty} dp' \underbrace{\left[\frac{1}{p'^2-q^2\mp i\delta}\, p'\right]}_{\frac{1}{2} \text{ for any pole}}\sin p'r$$

↪ $\underline{\mathrm{Im}}\, e^{ip'r}$

↳ take it outside

$$= \frac{1}{4\pi^2}\frac{1}{r}\,(2\pi)\,\frac{1}{2}\, e^{\pm iqr}$$

$$= \frac{1}{4\pi r}\, e^{\pm iqr}$$

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