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①

Free Scalar Fields

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

Eqn. of Motion (EOM)

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} = \frac{\partial \mathcal{L}}{\delta \varphi}$$

$$(\partial^2 + m^2)\,\varphi_x = 0$$

$$\pi_x := \frac{\delta \mathcal{L}}{\delta \partial_t \varphi} = \partial_t \varphi_x$$

$$[\varphi_x, \pi_{x'}] = i\,\delta^3_{\vec{x}-\vec{x}'} \quad \text{if} \quad t = t'$$

equal time commutation relation

(ETCR)

Realization # 1

$$\varphi_x = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_p}} \left(e^{-ip\cdot x} a_{\vec{p}} + e^{ip\cdot x} a^\dagger_{\vec{p}} \right)$$

$p^0 = \mathcal{E}_{\vec{p}}$ i.e. on-shell

$$\pi_x = \int \frac{d^3p}{(2\pi)^3} \, -i\sqrt{\frac{\mathcal{E}_p}{2}} \left(e^{-ip\cdot x} a_{\vec{p}} - e^{ip\cdot x} a^\dagger_{\vec{p}} \right).$$

$$\langle 0| a_{\vec{p}} \, a^\dagger_{\vec{p}'} |0\rangle = (2\pi)^3 \delta^3_{\vec{p}-\vec{p}'}$$

check that

$$\Rightarrow \quad [\varphi_{\vec{x}}, \pi_{\vec{y}}] = i\,\delta^3_{\vec{x}-\vec{y}}$$

$$\langle 0| T\{\varphi_x \varphi_y\} |0\rangle$$

$$= \theta(x^0-y^0) \langle 0| \varphi_x \varphi_y |0\rangle + \theta(y^0-x^0) \langle 0| \varphi_y \varphi_x |0\rangle$$

$$= \theta(x^0-y^0) \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_1}} \frac{1}{\sqrt{2\mathcal{E}_2}} e^{-ip_1\cdot x} e^{ip_2\cdot y} (2\pi)^3 \delta^3_{\vec{p}_1-\vec{p}_2} +$$

$$\theta(y^0-x^0) \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_1}} \frac{1}{\sqrt{2\mathcal{E}_2}} e^{ip_1\cdot x} e^{-ip_2\cdot y} (2\pi)^3 \delta^3_{\vec{p}_1-\vec{p}_2}$$

the 2 terms summarize nicely by

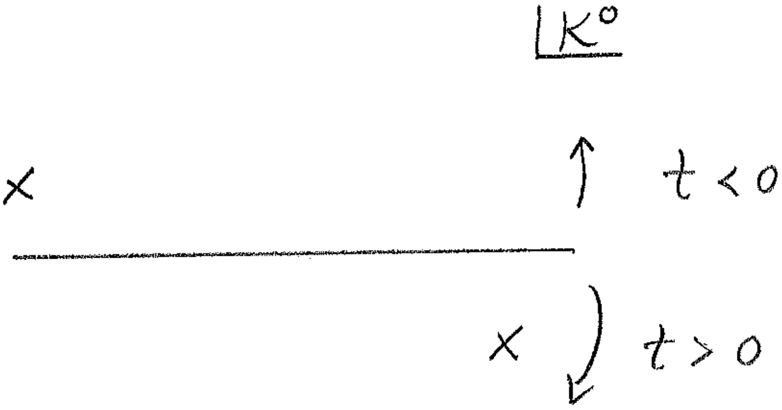
$$\to \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2E_k}\, e^{+i\vec{k}\cdot(\vec{x}-\vec{y}) - iE_k\underline{|x^0-y^0|}}$$

↳ check the sign

$$= \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\, e^{-ik\cdot(x-y)}.$$

$\updownarrow$

iG_2



$t = x^0 - y^0$

$$\mathcal{H} = \pi \dot{\varphi} - \mathcal{L}$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2$$

take $t \to 0$ $\quad$ $\mathcal{H}$ is t-independent anyway

$$H = \int d^3x \, \mathcal{H}_x$$

$$\langle 0 | H | 0 \rangle$$

easier to think of $-\varphi\nabla^2\varphi$

$$= V \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left(\frac{\mathcal{E}_p}{2} + \frac{\vec{p}^2 + m^2}{2\mathcal{E}_p} \right)$$

$$= V \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \mathcal{E}_p \quad // \qquad \text{vac. energy}$$

In fact, before sandwiched by $\langle 0 | \cdots | 0 \rangle$

$$\hat{H} = \int d^3x \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} e^{i(\vec{p}_1 + \vec{p}_2)\cdot\vec{x}} \frac{1}{2} \times$$

$$\left\{ -\sqrt{\frac{\mathcal{E}_1 \mathcal{E}_2}{4}} \left(a_{\vec{p}_1} - a^\dagger_{-\vec{p}_1}\right)\left(a_{\vec{p}_2} - a^\dagger_{-\vec{p}_2}\right) + \frac{1}{\sqrt{4\mathcal{E}_1\mathcal{E}_2}} \left(-\vec{p}_1\cdot\vec{p}_2 + m^2\right)\left(a_{\vec{p}_1} + a^\dagger_{-\vec{p}_1}\right)\left(a_{\vec{p}_2} + a^\dagger_{-\vec{p}_2}\right) \right\}.$$

$$\int d^3x \; e^{i(\vec{p}_1+\vec{p}_2)\cdot\vec{x}} \longrightarrow (2\pi)^3\,\delta^3_{\vec{p}_1+\vec{p}_2}.$$

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2} \times \Big\{$$

$$\left(-\frac{\varepsilon_p}{2}\right)\left(a_{\vec{p}}\,a_{-\vec{p}} - a_{\vec{p}}\,a^\dagger_{\vec{p}} - a^\dagger_{-\vec{p}}\,a_{-\vec{p}} + a^\dagger_{-\vec{p}}\,a^\dagger_{\vec{p}}\right) +$$

$$\left(\frac{\varepsilon_p}{2}\right)\left(a_{\vec{p}}\,a_{-\vec{p}} + a_{\vec{p}}\,a^\dagger_{\vec{p}} + a^\dagger_{-\vec{p}}\,a_{-\vec{p}} + a^\dagger_{-\vec{p}}\,a^\dagger_{\vec{p}}\right)\Big\}.$$

$$\leftrightarrow a^\dagger_{\vec{p}}\,a_{\vec{p}}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\,\varepsilon_p\left(a_{\vec{p}}\,a^\dagger_{\vec{p}} + a^\dagger_{-\vec{p}}\,a_{-\vec{p}}\right)$$

at the same $\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3}\,\varepsilon_p\left(\underline{a^\dagger_{\vec{p}}\,a_{\vec{p}}} + \frac{1}{2}\left[a_{\vec{p}}, a^\dagger_{\vec{p}}\right]\right)$$

$:\hat{H}:$ normal order.

" $(2\pi)^3\,\delta^3(\vec{p}-\vec{p})$. "

$\langle 0|:\hat{H}:|0\rangle \longrightarrow 0$

Where is the V factor?

$$(2\pi)^3 \delta^3_{\vec{p}-\vec{p}} \leftrightarrow V.$$

$$\langle 0|\hat{H}|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \varepsilon_p \quad V.$$

as obtained before!

$\hookrightarrow$

vac pressure:

$$P_{vac} = -\frac{\partial E}{\partial V}$$
$$= -\int \tfrac{1}{2}\varepsilon_p$$

Dark energy.

Similarly

$$\underline{\hat{\vec{p}}} \equiv \int d^3x \; -\pi(x) \nabla \varphi(x)$$

$$\longrightarrow \int \frac{d^3p}{(2\pi)^3} \vec{p} \, a^\dagger_{\vec{p}} a_{\vec{p}}$$

$$\int \frac{d^3p}{(2\pi)^3} \vec{p} \to \vec{0}$$
by sym.

L & S 1.19.

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\nu \varphi - \delta^\mu{}_\nu \mathcal{L}$$

$$\int T^{00} \leftrightarrow H$$

$$\int T^{0i} \leftrightarrow \underline{\hat{\vec{p}}}$$

Realization # 2.

$$\langle 0 | T\{\varphi_x \varphi_y\} | 0 \rangle = i G_2$$

$$G_2(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\delta} e^{-ik\cdot(x-y)}$$

$$(\partial_x^2 + m^2) G_2 = -\delta^4_{x-y}$$

or

$$\underline{(\partial_x^2 + m^2) \langle 0 | T\{\varphi_x \varphi_y\} | 0 \rangle} = -i \delta^4_{x-y}.$$

verify LHS directly :

$$(\partial_x^2 + m^2) \left[\theta_{x^0-y^0} \langle \varphi_x \varphi_y \rangle + \theta_{y^0-x^0} \langle \varphi_y \varphi_x \rangle. \right]$$

$$\partial_t^2 \left[\theta(x^0-y^0) \langle \varphi_x \varphi_y \rangle \right]$$

$$= \partial_t \left(\delta_{x^0-y^0} \langle \varphi_x \varphi_y \rangle + \theta(x^0-y^0) \langle \dot\varphi_x \varphi_y \rangle \right)$$

$$= \delta_{x^0-y^0} \langle \dot\varphi_x \varphi_y \rangle + \underline{\theta(x^0-y^0) \langle \ddot\varphi_x \varphi_y \rangle}$$

such term $\to 0$
when $-\nabla^2 + m^2$ acts

LHS

$$\delta(x^0-y^0)\quad \langle \dot\varphi_x \varphi_y - \varphi_y \dot\varphi_x \rangle$$

$$[\dot\varphi_x, \varphi_y] \rightarrow -i\,\delta^3_{\vec x-\vec y}\,.$$

at equal time

check the sign

$$(\partial_x^2+m^2)\langle T\{\varphi_x\varphi_y\}\rangle = -i\,\delta^4_{x-y}$$

$$\Rightarrow \quad \langle T\{\varphi_x\varphi_y\}\rangle = \frac{-i}{\partial_x^2+m^2}\,\delta^4_{x-y}$$

$$= \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\delta}\,e^{-ik\cdot(x-y)}$$

//

Realization 3.

~~wait till Functional~~ approach...

wait is over!

$$Z(h) = \int_{-\infty}^{+\infty} dx \; e^{-x^2 + hx}$$

$$= \sqrt{\pi} \; e^{\frac{h^2}{4}}$$

$$\frac{1}{Z}\frac{\partial Z}{\partial h} = \frac{\int dx \; x \; e^{-x^2+hx}}{\int dx \; e^{-x^2+hx}}$$

$$\frac{\partial \ln Z}{\partial h} \longrightarrow \langle x \rangle (h).$$

$$\frac{\partial^2}{\partial h \partial h} \ln Z = \frac{\partial}{\partial h}\left(\frac{1}{Z}\frac{\partial Z}{\partial h}\right)$$

$$= \frac{1}{Z}\frac{\partial^2 Z}{\partial h \partial h} - \left(\frac{1}{Z}\frac{\partial Z}{\partial h}\right)^2$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

$$= \langle x^2 \rangle_c .$$

$$\ln Z \sim \frac{1}{4} h^2 + C .$$

$$\langle x^2 \rangle_c \mapsto \frac{1}{2}$$

take $h \to 0$
at the end.

$$Z(\vec{h}) = \int dx_1 dx_2 \dots dx_N \; e^{-\vec{x}^t A \vec{x} + \vec{h}^t \vec{x}}$$

$$= \frac{\sqrt{\pi}^N}{\sqrt{\det A}} \; e^{\frac{1}{4}\vec{h}^t A^{-1} \vec{h}} \quad //$$

$$Z[j_x] = \int D\varphi \; e^{i\int \left[-\frac{1}{2}\varphi_x(\partial^2+m^2)\varphi_x + j_x\varphi_x\right]}$$

$$\langle \varphi_x \rangle = \frac{1}{Z}\frac{-i\,\delta Z}{\delta j_x} = \frac{\int D\varphi\; \varphi_x\, e^{i\int \mathcal{L}+j\varphi}}{\int D\varphi\; e^{i\int \mathcal{L}+j\varphi}}$$

$$\langle 0|\, T\{\varphi_x \varphi_y\}\,|0\rangle_c = (-i)^2\frac{\delta^2}{\delta j_x \delta j_y}\ln Z$$

$$= \frac{\int D\varphi\; \varphi_x\varphi_y\; e^{i\int \mathcal{L}+j\varphi}}{\int D\varphi\; e^{i\int \mathcal{L}+j\varphi}} - \langle\varphi_x\rangle\langle\varphi_y\rangle$$

$$Z \to \frac{1}{\sqrt{\det(\partial^2+m^2)}}\; e^{\frac{\int_x j_x \frac{-1}{4\, i\frac{1}{2}(\partial_x^2+m^2)} j_x}{}}$$

$$\updownarrow$$

$$i\frac{1}{2}\int_{xy} j_x \frac{1}{\partial_x^2+m^2}\delta^4_{xy}\, j_y$$

$$-\frac{\delta^2}{\delta j_x \delta j_y}\ln Z = -i\,\frac{1}{\partial_x^2+m^2}\delta^4_{xy} = i\,G_{xy}$$

$$\updownarrow$$

$$\langle T\{\varphi_x\varphi_y\}\rangle$$