03 . 2025

Running Coupling

①

QED

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \ln \frac{Q^2}{\mu^2}}$$ Landau formula

$$\alpha^{-1}(Q^2) = \alpha^{-1}(\mu^2) - \frac{1}{3\pi} \ln \frac{Q^2}{\mu^2}$$

$$Q^2 \frac{d}{dQ^2} \alpha^{-1} = \frac{-1}{\alpha^2} Q^2 \frac{d}{dQ^2} \alpha = \frac{-1}{3\pi}$$

$$\Rightarrow \qquad \beta_\alpha := Q^2 \frac{d}{dQ^2} \alpha = \frac{1}{3\pi} \alpha^2$$ PDG convention

we see that $\beta_{QED} > 0$ $\alpha \uparrow$ when $Q \uparrow$

If we define Λ_{QED} s.t

$$- \frac{\alpha(\mu^2)}{3\pi} \ln \frac{Q^2}{\Lambda^2_{QED}} = 1 - \frac{\alpha(\mu^2)}{3\pi} \ln \frac{Q^2}{\mu^2}$$

$$\alpha(Q^2) \longrightarrow \frac{-3\pi}{\ln \frac{Q^2}{\Lambda^2_{QED}}}$$

makes sense only for $Q^2 < \Lambda^2_{QED}$

$\alpha \to \infty$ as $Q \to \Lambda_{QED}$

$$\Lambda^2_{QED} = \mu^2 \underline{e^{\frac{3\pi}{\alpha(\mu^2)}}}$$

$\hookrightarrow$ generally : $e^{-\int^{\alpha(\mu^2)} d\alpha' \frac{1}{\beta(\alpha')}}$

Λ_{QED} : the dependence on μ is an illusion !

$$\mu^2 \frac{d}{d\mu^2} \Lambda_{QED} = 0 = \underbrace{\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta \frac{\partial}{\partial \alpha} \right)}_{\text{they cancel}} \Lambda_{QED}.$$

$\Lambda_{QED}(\mu, \alpha(\mu))$ s.t.

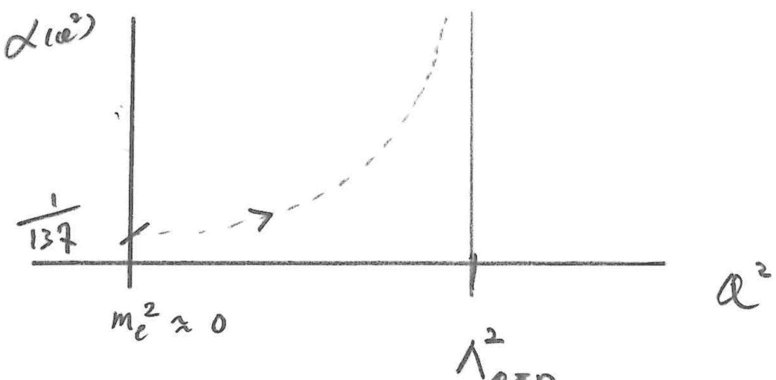
$\frac{d}{d\mu} \Lambda_{QED} = 0$ —— Renormalization Group.

$\beta_{QED} > 0$ $\quad Q$ is at most Λ_{QED}

at UV, the only theory allowed are free theory $\longrightarrow$ asymptotic trivial

$\neq$ asymptotic free !

$\Lambda_{QED} \leftrightarrow$ Landau pole



take $\mu \to m_e$ & $\alpha(m_e^2) = 1/137$

$\Lambda_{QED} \sim 10^{280}$ MeV. Good luck !

QCD

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\,\beta_0 \ln \frac{Q^2}{\mu^2}}$$

Gross – Politzer – Wilczek formula

where

$$\beta_0 = \frac{33 - 2N_f}{12\pi}$$

Again define Λ_{QCD} via

$$\alpha_s(\mu^2)\,\beta_0 \ln \frac{Q^2}{\Lambda_{QCD}^2} = 1 + \alpha_s(\mu^2)\,\beta_0 \ln \frac{Q^2}{\mu^2}$$

$$\alpha_s(Q^2) \longrightarrow \frac{1}{\beta_0 \ln \frac{Q^2}{\Lambda_{QCD}^2}}$$

$$\Lambda_{QCD}^2 = \mu^2\, e^{-\frac{1}{\beta_0 \alpha_s(\mu^2)}}$$

$\alpha_s(Q^2)$ makes sense only for $Q^2 \gg \Lambda_{QCD}^2$.

$$\beta_{QCD} := Q^2 \frac{d}{dQ^2} \alpha_s = -\beta_0\, \alpha_s^2$$

$\Rightarrow$ $\beta_{QCD} < 0$ is Noble Prize Worthy!

$Q \uparrow$ $\alpha_s \downarrow$ $\Rightarrow$ Asymptotic Freedom

Asymptotic freedom is for UV

nothing to do w Confinement → IR structure

→ strategy in solving flow equ

$$Q^2 \frac{d}{dQ^2}\alpha = \beta_\alpha = \frac{1}{3\pi}\alpha^2$$

$$t := \ln Q^2$$

$$\frac{d}{dt}\alpha = \frac{1}{3\pi}\alpha^2 \, z_f$$

eff. flavor

1). for QED :

Initial vals.

$$t \to \ln \frac{Q^2}{M_e^2} \quad : \quad t = 0 \to \ln \frac{M_Z^2}{M_e^2}$$

$$\alpha(t=0) \to 1/137$$

$\alpha(t)$ can be solved

effective flavor : at $Q \to M_Z$

Active massless flavor → e, μ, τ ; u, d, s, c, b

$$z_f = 1 + 1 + 1 + 2 \times N_c \times \left(\tfrac{2}{3}\right)^2 +$$

$3 \times N_c \times \ldots$

b). for QCD :

~~#~~ flavor at M_Z : 5 flavors

$N_f \to 5$

coeffs depends on N_f

$$\frac{d\alpha}{dt} = \beta(\alpha)$$

here we have no good "initial value" as the formula X work at low Q^2

strategies :

① Q_1 is defined s.t. $\alpha(Q_1^2) = 1$

we tune μ_{ref} → to get

$$\alpha(M_Z^2) = 0.118 \quad \text{exp. val.}$$

$$\to Q_1 \sim 425 \text{ MeV}$$

$N_f = 5$
4-loop

below Q_1 QCD is non-perturbative ($\Lambda_{QCD} < Q_1$)

② integrates backward from $Q^2 = M_Z^2$

note :

$$\alpha_{\,Q^2 \to \Lambda_{QCD}^2} \to \infty$$

↳ ~ 200 MeV

Λ_{QCD} is a natural scale

to measure obs.

e.g. $V \sim E \sim \frac{1}{r} f(\Lambda_{QCD} r)$

$$\sim \underline{\sigma}\, r$$

$\hookrightarrow$ measured

$\sim \Lambda^2_{QCD}$.