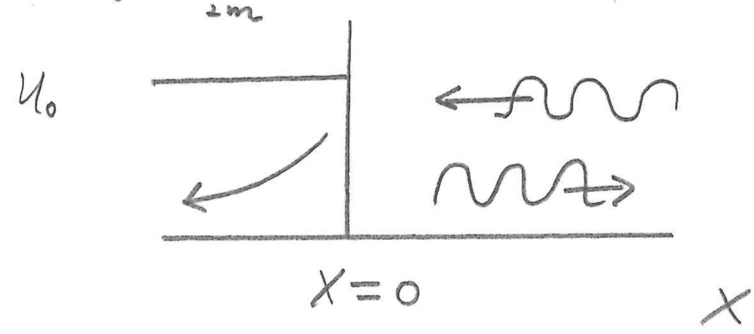
time delay

$$U_0 - E = \frac{k'^2}{2m}$$



$$E(k) = \frac{k^2}{2m} < U_0$$

$$\psi(x) = \begin{cases} A_2\, e^{-k'|x|} & \text{if } x < 0 \\ A_1 \sin(kx + \delta_k) & \text{if } x > 0 \end{cases}$$

$$k \to \sqrt{2mE}$$

$$k' \to \sqrt{2m(U_0 - E)}$$

$\{A_1, A_2, \delta\} \longrightarrow$ 2 continuity conditions at $x = 0$

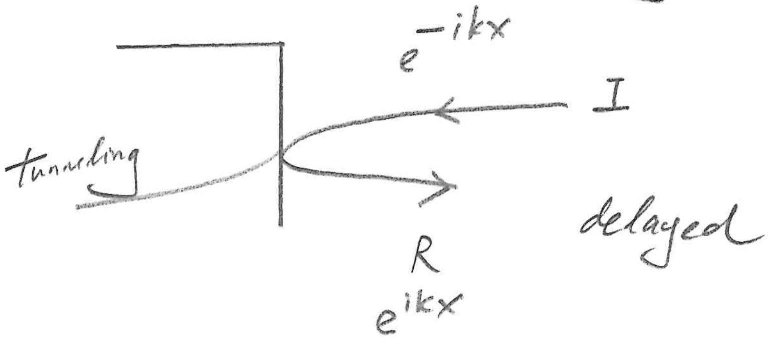
$$\begin{aligned} A_1 \sin\delta &= A_2 \\ k A_1 \cos\delta &= k' A_2 \end{aligned} \Rightarrow \tan\delta = \frac{k}{k'} = \frac{\sqrt{E}}{\sqrt{U_0 - E}}$$

$$\& \quad \frac{A_2}{A_1} = \sin\delta$$

$$\psi_{x>0} \to \frac{1}{2} i A_1 e^{-i\delta_k}\left(e^{-ikx} - e^{ikx+2i\delta_k}\right)$$

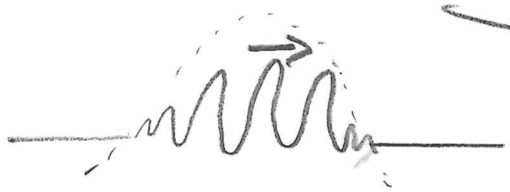
Incoming

Reflected



Group velocity vs phase velocity

wave packet



$$\psi_{x,t} \sim \int \frac{dk'}{2\pi} \underline{f_{k'}}\, e^{ik'x} e^{-iE_{k'}t}$$

modulation.

$f_{k'} \to$ center around k_0.

$k' \simeq k_0 + \Delta k'$

$$\psi_{x,t} \sim e^{i\left(x - \frac{E_{k_0}}{k_0}t\right)} \int \frac{d\Delta k'}{2\pi} f_{k_0+\Delta k'}\, e^{i\Delta k'\left(x - \frac{\partial E_k}{\partial k} t\right)}$$

phase velocity : v_p

v_{gp}.

how packet moves

$$E_k = \frac{k^2}{2m}$$

$$v_p \to \frac{k}{2m} \qquad v_{gp} \to \frac{k}{m} = 2 v_p$$

↑ the true velocity !

Back to the scattering problem

$$\psi_{x,t} \sim \int \frac{dk''}{2\pi} f_{k''} \left(e^{-ik''x} - e^{ik''x + 2i\delta_{k''}} \right) e^{-iE_{k''}t}$$

$$v_{gp} \to \left. \frac{\partial E_{k''}}{\partial k''} \right|_{k''=k}$$

$$\langle x \rangle_I = - v_{gp} t$$

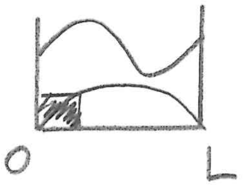
$$\langle x \rangle_R \to \frac{\partial E_k}{\partial k} t - 2 \frac{\partial \delta_k}{\partial k}$$

$$= \frac{\partial E}{\partial k} \left(t - 2 \frac{1}{\frac{\partial E_k}{\partial k}} \frac{\partial \delta_k}{\partial k} \right)$$

$$= \frac{\partial E}{\partial k} \left(t - \underbrace{2 \frac{\partial \delta}{\partial E}}_{\text{time delay}} \right)$$

reaches origin not at $t=0$ but at $t = 2 \frac{\partial \delta}{\partial E}$

time delay $\longleftrightarrow$ DoS



$$\psi^{(0)}_{(x)} = \sin kx$$

$$k_n^{(0)} = \frac{n\pi}{L} \qquad n = 1, 2, 3, \dots$$

With an obstacle:

$$\psi_{(x)} \longrightarrow \sin(kx + \delta_k)$$

BC

$$k_n L + \delta_{k_n} = n\pi \qquad n = 1, 2, \dots$$

$$\frac{dN}{dk} \longrightarrow \frac{L}{\pi} + \underline{\frac{1}{\pi}\frac{d\delta_k}{dk}}$$

state per k
//
DoS

Non-int
DoS

Δ in DoS
due to interaction

$$\frac{d\Delta N}{dE} \longrightarrow \frac{1}{2\pi}\left(2\frac{d\delta}{dE}\right)$$

3D

$$dN_k = \frac{1}{8}\left(\frac{L}{\pi}\right)^3 d^3k$$

Walecka
QM
problem 7.6

— Fermi Gas in a box

for scat $\rightarrow$ PBC

$$e^{ikx}$$

$$k \rightarrow \frac{2\pi}{L} n_x$$

$$n_x \rightarrow 0, \pm 1, \pm 2, \ldots$$

$$dN = \frac{L}{2\pi} dk$$

DOS

$$\rho^{1D}(E) \rightarrow \frac{dN}{dE} = \frac{L}{2\pi} \frac{dk}{dE}$$

$$\rightarrow \frac{L}{2\pi} \frac{m}{k}$$

$$k = \sqrt{2mE}$$

3D

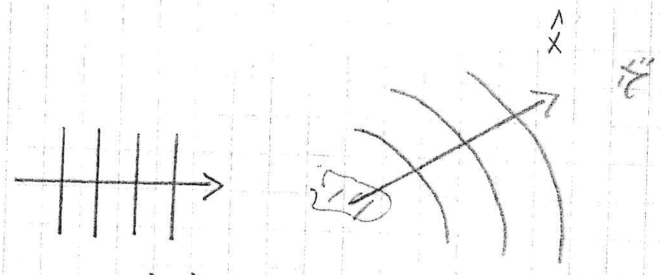
$$dN \rightarrow \left(\frac{L}{2\pi}\right)^3 d^3k$$

$$\rightarrow V \frac{d^3k}{(2\pi)^3}$$

$$N_{\text{Fermi Gas}} \rightarrow \underset{\text{spin degen}}{2} V \int^{k_F} \frac{d^3k}{2\pi^3}$$

$$= V \frac{1}{3\pi^2} k_F^3$$

Scattering Theory



$$E = \frac{q^2}{2m_R}$$

$e^{i\vec q\cdot\vec r}$ $\qquad\qquad$ $\frac{e^{iqr}}{r} f_q$

$$\psi_{\vec x} \xrightarrow{r\to\infty} e^{i\vec q_i\cdot\vec x} + \frac{e^{iqr}}{r} f_{q}$$

$$\vec q_f = q\hat x$$

$$H|\psi\rangle = (H_0+V)|\psi\rangle = E|\psi\rangle$$

$$(E-H_0)|\psi\rangle = V|\psi\rangle$$

$$\Rightarrow |\psi\rangle = |\varphi\rangle + \frac{1}{E-H_0}V|\psi\rangle$$

satisfies

$$(E-H_0)|\varphi\rangle = 0$$

$$\langle\vec x|\psi\rangle = \langle\vec x|\varphi\rangle + \langle\vec x|\frac{1}{E-H_0}V|\psi\rangle$$

$$e^{i\vec q_i\cdot\vec x}$$

$$\int d^3x' \langle\vec x|\frac{1}{E-H_0}|\vec x'\rangle V(\vec x')\psi(\vec x')$$

$$r\to\infty$$

$$\Leftrightarrow \frac{1}{r}e^{iqr} f$$

$$G^0_{\vec{x}\vec{x}'} = \langle \vec{x} | \frac{1}{E - H_0 + i\delta} | \vec{x}' \rangle$$

↳ to properly define the Green's function

$$H_0 = \frac{-\nabla^2}{2M_R}$$

$$G^0_{\vec{x}\vec{x}'} = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{E - \frac{\vec{p}'^2}{2M_R} + i\delta} e^{i\vec{p}'\cdot(\vec{x}-\vec{x}')}$$

$$= 2M_R \frac{-1}{4\pi|\vec{x}-\vec{x}'|} e^{iq|\vec{x}-\vec{x}'|}$$

$$\parallel$$

$$\psi_{\vec{x}} = \varphi_{\vec{x}} + \int d^3x' \frac{-2m_R}{4\pi} \frac{e^{iq|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} V_{\vec{x}'} \psi_{\vec{x}'}$$

$r \to \infty$

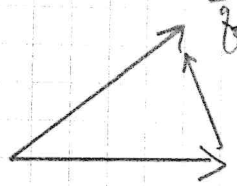
$$\longrightarrow \varphi_{\vec{x}} + \frac{e^{iqr}}{r} \underline{\int d^3x' \frac{-2m_R}{4\pi} e^{-iq\hat{x}\cdot\vec{x}'} V_{\vec{x}'} \psi_{\vec{x}'}}$$

f_q is directly identified

Born Approx

replace $\psi_{\vec{x}'}$ in the integral with $\phi_{\vec{x}'} = e^{i\vec{q}_i\cdot\vec{x}'}$

$$f_q^{Born} = \frac{-2m_R}{4\pi}\int d^3x' \, e^{-i\Delta\vec{q}\cdot\vec{x}'} \, V_{\vec{x}'}$$



$\vec{q}_f = q\hat{x}$

$\Delta\vec{q} = \vec{q}_f - \vec{q}_i$

$\vec{q}_i$

$$\longrightarrow \frac{-2m_R}{2\pi} \, \tilde{V}_{\Delta\vec{q} = \vec{q}_f - \vec{q}_i}$$

$f_q \leftrightarrow$ − of the Fourier transform of the potential

$$f_q = \frac{-2m_R}{4\pi}\int d^3x' \, e^{-iq\hat{x}\cdot\vec{x}'} \, V_{\vec{x}'} \, \psi_{\vec{x}'}$$

remains a general way to compute f_q

Another View : T-matrix

$$|\psi\rangle = |\varphi\rangle + \frac{1}{E-H_0} V |\psi\rangle$$

$$V|\psi\rangle = V|\varphi\rangle + V\frac{1}{E-H_0}V|\psi\rangle$$

$$= V|\varphi\rangle + V\frac{1}{E-H_0}V|\varphi\rangle +$$

$$V\frac{1}{E-H_0}V\frac{1}{E-H_0}V|\varphi\rangle + \cdots$$

$$= \left\{ V + V\hat{G}_0 V + V\hat{G}_0 V \hat{G}_0 V + \cdots \right\} |\varphi\rangle$$

$$\updownarrow$$

$$T := \frac{V}{1 - G_0 V} = V + V G_0 T$$

$$G_0^{\pm} = \frac{1}{E - H_0 \pm i\delta}$$

$$\Rightarrow V|\psi\rangle = T|\varphi\rangle$$

T works on std. basis $|\varphi\rangle$
& contain all info. in $|\psi\rangle$

$T = V + V G_0 T \longrightarrow$ Lippmann - Schwinger eq^n.

$\Rightarrow$ get rid of $|\psi\rangle$, $|\varphi\rangle$

this yields another way to compute f_q :

$$\psi_{\vec{x}} = \varphi_{\vec{x}} + \int d^3x' \, G^0_{\vec{x}\vec{x}'} \langle \vec{x}' | T | \vec{q}_i \rangle$$

$r \to \infty$

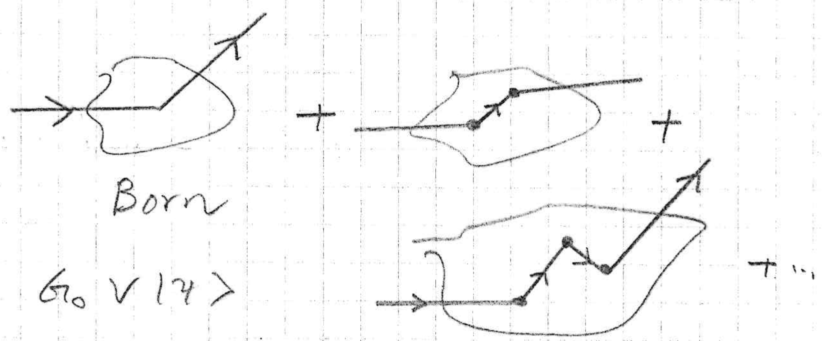
$$\to \varphi_{\vec{x}} + \frac{e^{iqr}}{r} \underbrace{\langle \vec{q}_f | T | \vec{q}_i \rangle \left(\frac{-2M_R}{4\pi} \right)}_{f_q}$$

4 $T \to V$

$\Longleftrightarrow$

Born approx.

T-matrix T stands for transfer matrix



higher order term

$$|\psi\rangle = |\varphi\rangle + G_0 V |\psi\rangle$$
$$= (I + G_0 T)|\varphi\rangle$$

$|\psi\rangle \leftrightarrow |\varphi\rangle$

translation

Yet another view : resolvent G

$$G = \frac{1}{E-H}$$

$$G^0 = \frac{1}{E-H_0}$$

$$G^{-1} = G_0^{-1} - V$$

$$G = \frac{1}{G_0^{-1} - V}$$

$$= G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots$$

$$= G_0 + G_0 T G_0$$

see $VG = TG_0$?

$$G = G_0 + G_0 T G_0$$

$\downarrow$

full Green's function

e.g.

$\rho_E = -2 \operatorname{Im} G$ to study spectral function

$\mathrm{Im}\, T \neq 0$ even when $\mathrm{Im}\, V = 0$

how $\mathrm{Im}\, T$ is generated ?

T (or f) has to be complex to satisfy optical theorem

$$f = \frac{1}{q} e^{i\delta} \sin\delta$$

$$\Rightarrow \frac{1}{q} \mathrm{Im} f = |f|^2$$

$\mathrm{Im}\, G_0 \simeq 0$ s.t. $\mathrm{Im}\, G \neq 0$

$\longrightarrow$ Width ?!

How ?

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \ldots$$

---

resum the series to get a width

$$\mathrm{Im}\, G_0^+ \longrightarrow -\pi \delta_{E-\hat{H}_0}$$

$$-2 \,\mathrm{tr}\, \mathrm{Im}\, G_0^+ \rightarrow \mathrm{tr}\, 2\pi \delta_{E-\hat{H}_0}$$

$\longleftrightarrow$ ρ_{NR} phase space

$$\rho_{NR} = \int \frac{d^3 q}{(2\pi)^3} 2\pi \delta_{E-\frac{q^2}{2m_R}}$$

$$= \frac{m_R q}{\pi}$$

Summary

$H_{int} \rightarrow$ T-matrix

$$f_q = \frac{-2m_e}{4\pi} \langle \vec{q}_f | T | \vec{q}_i \rangle$$

$$T = V + V G_0 T$$

$\angle S$
q^2.

a natural − sign
between T & f

$$f_q \leftrightarrow \frac{1}{q} \sin\delta \, e^{i\delta}$$

or

$$S = e^{2i\delta} = 1 + 2iq f_q$$

$$= 1 + i \frac{2\pi \, \ell_{NR}}{m_R} f_q$$

$$= 1 - i \, \ell_{NR} \, T$$

it can be generalized:
see scat theory notes

$$\hat{S} = I - i \, 2\pi \, \delta_{E-H_0} \, \hat{T}$$

$$e^{2i\delta} \leftrightarrow 1 - i \ell T \leftrightarrow 1 + 2iqf$$

$$\ell \rightarrow \frac{q}{4\pi\sqrt{s}} \qquad T = -8\pi\sqrt{s} f$$

convention :

Scattering theorist need

connect to LS

$$\hat S = I - i\, 2\pi\, \delta_{E-H_0}\, \hat T_{scat}$$

$$= \underbrace{G_0^* \, G^{*-1}}_{\Omega_2^{-1}} \; \underbrace{G \, G_0^{-1}}_{\Omega_1}$$

$$G_0 = \frac{1}{E - H_0 + i\delta}$$

$$G = \frac{1}{E - H + i\delta}$$

$$\mathrm{tr} \ln \hat S = \ln \left[1 - i \, e \, \langle \hat T_{scat} \rangle \right]$$

QFT people need ...

$$\hat S = I + i (2\pi)^4 \delta^4_{P_f P_i} \, M_{QFT}$$

$$i M_{QFT} \leftrightarrow \text{Feynman rules}$$

$$M_{QFT} \leftrightarrow - \hat T_{scat}$$

is unfortunate !!

self energy check

$$\Sigma \sim -4\pi f\, n$$

$$f \simeq \frac{-1}{4\pi} 2m_R T$$

$$\Rightarrow \Sigma \sim 2m_R T\, n$$

if $V < 0$ attractive int.

$f > 0$ $\quad f \underset{g\to 0}{\simeq} a_{scat} > 0$

$\Sigma < 0$

Good convention to follow

Appendix   Quick way to $G^{\pm}_{\vec{x}}$:

$$G^{\pm} = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{E - \frac{p'^2}{2m_R} \pm i\delta} e^{i\vec{p}'\cdot\vec{x}}$$

$$= -2m_R \int \frac{d^3p'}{(2\pi)^3} \frac{1}{p'^2 - q^2 \mp i\delta}$$

using

$$\int \frac{d^3p'}{(2\pi)^3} \frac{1}{p'^2 + m^2} e^{i\vec{p}'\cdot\vec{x}} = \frac{1}{4\pi r} e^{-\sqrt{m^2}\, r}$$

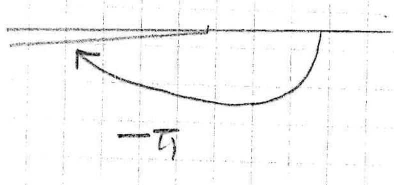
$$G^{\pm} \longrightarrow -2m_R \frac{1}{4\pi r} e^{-\sqrt{-q^2 \mp i\delta}\; r}$$

now

$$\sqrt{-q^2 \mp i\delta} = \mp i q$$

why?

e.g. $\quad -q^2 - i\delta \quad \Longrightarrow$



$$\simeq q^2 e^{-i\pi}$$

$$\sqrt{} \longrightarrow q e^{-i\frac{\pi}{2}} = -iq$$

$$G^{\pm} = \frac{-2m_R}{4\pi} \frac{1}{r} e^{\pm i q r} \quad //$$

what if I don't want to use

FT $\frac{1}{p'^2+m^2} \leftrightarrow \frac{1}{4\pi r}e^{-mr}$

contour integral can be done directly

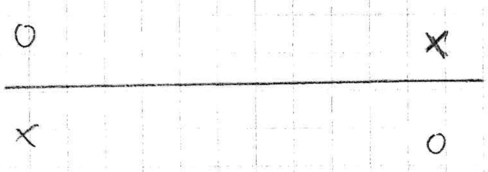
$$\int \frac{d^3p'}{(2\pi)^3}\,\frac{1}{p'^2-q^2\mp i\delta}\, e^{i\vec{p}'\cdot\vec{x}}$$

$$= \frac{2\pi}{8\pi^3}\int_0^\infty dp'\, \frac{1}{p'^2-q^2\mp i\delta}\, p'^2\, \frac{2\sin p'r}{p'r}$$

$$= \frac{1}{4\pi^2}\frac{1}{r}\int_{-\infty}^{+\infty} dp'\, \underbrace{\left[\frac{1}{p'^2-q^2\mp i\delta}\, p'\right]}_{\text{gives } \frac{1}{2} \text{ for any pole}} \sin p'r$$

$\to \underline{\mathrm{Im}}\; e^{ip'r}$

↳ take it outside



has to close up

$$= \frac{1}{4\pi^2}\frac{1}{r}\,(2\pi)\,\frac{1}{2}\, e^{\pm iqr}$$

$$= \frac{1}{4\pi r}\, e^{\pm iqr}$$

//