Introduction to Schwinger Dyson Equations

$$Z(h) = \int_{-\infty}^{+\infty} dx \; e^{-x^2+hx}$$

$$= \sqrt{\pi} \; e^{h^2/4}$$

$$\langle x \rangle (h) = \frac{1}{Z} \int dx \; x \; e^{-x^2+hx}$$

$$= \frac{\partial}{\partial h} \ln Z$$

$$0 = \int dx \frac{d}{dx} e^{-x^2+hx}$$

$$= \int dx \, (-2x+h) \, e^{-x^2+hx}$$

$$0 = \left(-2 \frac{\partial}{\partial h} + h\right) Z(h)$$

$$\swarrow \quad x \leftrightarrow \frac{\partial}{\partial h}$$

$$Z(h) = Z(0) \, e^{h^2/4}$$

$$\hookrightarrow \sqrt{\pi}$$

(functional)

In QFT → only the differential eqns are well-defined

one more trick

$$Z(h) = \int dx \; e^{-S(x) + hx}$$

DSE

$$0 = \int dx \; (-S' + h) \, e^{-S(x) + hx}$$

$$0 = \left(-S'_{x \to \frac{\partial}{\partial h}} + h\right) Z$$

$Z = e^{W(h)}$ $\quad$ W → generating function for connected f's.

$\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2$

$$\frac{\partial}{\partial h}\left(e^W f\right) = e^W W' f + e^W f'$$

$$= e^W \left(W' + \frac{\partial}{\partial h}\right) f$$

$$\Rightarrow \quad 0 = \left(-S'_{x \to W' + \frac{\partial}{\partial h}} + h\right) I = 0.$$

e.g. $\quad W = \frac{h^2}{4} + C$

one more

$W(h) \qquad X(h) = \frac{\partial W}{\partial h}$

$\Gamma(x) \qquad$ s.t $\qquad h = \frac{-\partial \Gamma}{\partial x}$

$h(x) \rightarrow$ inverting $X(h)$.

$$\Gamma = W - hx$$

$$\frac{\partial \Gamma}{\partial x} = \cancel{\frac{\partial W}{\partial h}} \frac{\partial h}{\partial x} - h - \cancel{x \frac{\partial h}{\partial x}}$$

$$\rightarrow -h$$

$$\frac{-\partial^2 \Gamma}{\partial x \partial x} = \frac{\partial h}{\partial x} = \left(\frac{\partial^2 W}{\partial h \partial h}\right)^{-1}$$

$$\frac{\partial}{\partial h_{(x)}} f = \int dx' \, \frac{\partial x'}{\partial h_{(x)}} \frac{\partial}{\partial x'} f$$

$$= \int dx' \left(\frac{-\partial^2 \Gamma}{\partial x \partial x'}\right)^{-1} \frac{\partial}{\partial x'} f$$

$$0 = -S'(x) + \frac{-\partial \Gamma}{\partial x}$$

$$x \rightarrow x + \int dx' \underbrace{\left(\frac{-\partial^2 \Gamma}{\partial x \partial x'}\right)^{-1}}_{G(x,x')} \frac{\partial}{\partial x'}$$

2-pt. function.

for our simple case

$$0 = -2x - \frac{\partial \Gamma}{\partial x}$$

$$\Rightarrow \Gamma = -x^2 + C$$

effective action is given by the action.

$$Z = \int dx \; e^{-x^2 + hx} = e^{W(h)} = e^{\Gamma + h(x) x}$$

$$= e^{-x^2 + \frac{-\partial \Gamma}{\partial x} x}$$

Summary

SDE $\quad Z_h = \int dx \; e^{-S(x) + hx}$

GE. $\quad \left(-S'_{x \to \frac{\partial}{\partial h}} + h\right) Z = 0$

connected GE $\quad \left(-S'_{x \to W' + \frac{\partial}{\partial h}} + h\right) \mathbb{I} = 0.$

1PI GEs $\quad -S'_{x \to x + \int dx' \, G_{xx'} \frac{\partial}{\partial x'}} \mathbb{I} - \frac{\partial \Gamma}{\partial x} = 0$

Application : φ^4 theory

$$\mathcal{L}_0 = \frac{1}{2}(\partial_x \varphi_x)^2 - \frac{1}{2} m^2 \varphi_x^2$$

$$(\partial_x^2 + m^2)\,\varphi_x = 0$$

eqn. of motion

$$\langle 0 | T\{\varphi_x, \varphi_y\} | 0 \rangle = i\, G_{xy}$$

$$(\partial_x^2 + m^2)\,\langle 0 | T\{\varphi_x \varphi_y\} | 0 \rangle = -i\, \delta^4_{xy}$$

$$\partial_t^2 \quad \left(\theta_{x^0-y^0} \langle \varphi_x \varphi_y \rangle + \theta_{y^0-x^0} \langle \varphi_y \varphi_x \rangle \right)$$

$$\rightarrow \quad \delta_{x^0-y^0} \quad \langle \underbrace{[\dot{\varphi}_{\vec{x}}, \varphi_y]}_{-i\,\delta^3_{\vec{x}-\vec{y}}} \rangle$$

$$(\partial_x^2 + m^2)\, \underline{G_{xy}} = -\delta^4_{xy}$$

$$G_p = \frac{1}{p^2 - m^2}$$

set up convention !

sign checked !

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

$$Z = \int D\varphi \; e^{i\int(\mathcal{L} + j\varphi)}$$

$$= e^{iW[j]}.$$

Z is the generating functional for Green's functions.

W is " for connected G.F.s.

SDE :

$$0 = \int D\varphi \frac{\delta}{\delta\varphi} e^{i\int(\mathcal{L} + j\varphi)}$$

$$0 = \int D\varphi \left[-(\partial^2+m^2)\varphi_x - \frac{\lambda}{6}\varphi_x^3 + j_x\right] e^{i\int(\mathcal{L}+j\,)}$$

$$\varphi_x \to -i\frac{\delta}{\delta j_x}$$

$$\left\{-(\partial^2+m^2)\varphi_x - \frac{\lambda}{6}\varphi_x^3 + j_x\right\} Z_j = 0.$$

$$\varphi_x \to -i\frac{\delta}{\delta j_x}$$

——— SDE #1.

$$\varphi_x^3 \longrightarrow \left(-i\frac{d}{dj_x}\right)^3$$

to go to the W-pic.

$$-i\frac{d}{dj_x}\left(e^{iW}A\right)$$

$$= e^{iW}\left(\frac{dW}{dj_x} - i\frac{d}{dj_x}\right)A.$$

recipe

$$-i\frac{d}{dj_x} \longrightarrow \left(\frac{dW}{dj_x} - i\frac{d}{dj_x}\right).$$

$$\left(-(\partial^2+m^2)\varphi_x - \frac{\lambda}{6}\varphi_x^3 + j_x\right)Z = 0$$

$\swarrow$

$$-(\partial^2+m^2)\frac{dW}{dj_x} - \frac{\lambda}{6}\underline{\left(\frac{dW}{dj_x} - i\frac{d}{dj_x}\right)^3 \mathbb{I}} + j_x = 0$$

horrible!

— SDE #2.

$$\left(\frac{\delta W}{\delta j} - i\frac{\delta}{\delta j}\right)^3 \mathbb{I}$$

$$= \left(\frac{\delta W}{\delta j} - i\frac{\delta}{\delta j}\right)\left(\frac{\delta W}{\delta j} - i\frac{\delta}{\delta j}\right)\frac{\delta W}{\delta j}$$

$$= \left(\frac{\delta W}{\delta j} - i\frac{\delta}{\delta j}\right)\left(\left(\frac{\delta W}{\delta j}\right)^2 - i\frac{\delta^2 W}{\delta j\,\delta j}\right)$$

$$= -\frac{\delta^3 W}{\delta j\,\delta j\,\delta j} + 3\,(-i)\frac{\delta W}{\delta j}\frac{\delta^2 W}{\delta j\,\delta j}$$

$$+ \left(\frac{\delta W}{\delta j}\right)^3$$

Now $\frac{\delta}{\delta j_y}$ on SDE #2.

$$-(\partial^2+m^2)\frac{\delta^2 W}{\delta j_x\,\delta j_y} - \frac{\lambda}{6}\frac{\delta}{\delta j_y}\left\{\frac{-\delta^3 W}{\delta j_x\,\delta j_x\,\delta j_x} + 3\,(-i)\frac{\delta W}{\delta j_x}\frac{\delta^2 W}{\delta j_x\,\delta j_x} + \left(\frac{\delta W}{\delta j_x}\right)^3\right\}$$

$$+ \delta^{(4)}_{xy} = 0$$

after $\frac{\delta}{\delta j} \longrightarrow j \to 0$ eventually.

odd ~~terms~~ vanish. w/o SSB

$$\frac{\delta}{\delta j_y}\{\cdots\} \longrightarrow \frac{-\delta^4 W}{\delta j_y\,\delta j_x\,\delta j_x\,\delta j_x} - 3i\,\frac{\delta^2 W}{\delta j_x\,\delta j_y}\,\frac{\delta^2 W}{\delta j_x\,\delta j_x} +$$

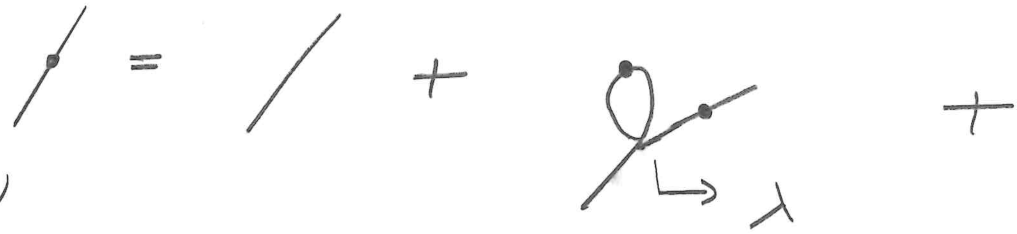
vanishing terms

recall

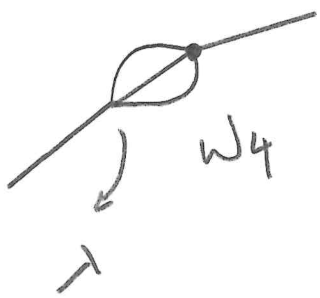
$$G_0 \longleftrightarrow \frac{-1}{\partial^2+m^2}\,\delta^{(4)}_{xy}$$

$$G_{x,y} \longleftrightarrow \frac{-d^2W}{dj_x\, dj_y} \implies \text{full 2-pt. function}$$

$$G_{xy} = G_{0\,xy} + \frac{\lambda}{2}\, i \int_z G_{0\,xz}\, G_{zy}\, G_{zz} + \frac{\lambda}{6}\int_z G_{0\,xz}\, W_{4\,zzzy}$$



$$G_{xy} = \left.\frac{-d^2W}{dj_x\, dj_y}\right|_{j\to 0}$$



$$W_4 \sim \int GGGGG$$

Γ - pic (advanced)

$$W = -i \ln Z$$

$$\frac{\delta^2 W}{\delta j_x \delta j_y} = -i\left(\frac{1}{Z}\frac{\delta^2 Z}{\delta j_x \delta j_y} - \frac{1}{Z}\frac{\delta Z}{\delta j_x}\frac{1}{Z}\frac{\delta Z}{\delta j_y}\right)$$

$$= i \langle T\{\varphi_x \varphi_y\}\rangle$$

$$= i \, i G_{xy} = -G_{xy}.$$

recall.

$$\langle T\{\varphi_x \varphi_y\}\rangle = i G_{xy}$$

$$Z = \int D\varphi \, e^{i\int(\mathcal{L} + j\varphi)}$$

$$\langle \varphi \rangle = -i \frac{\delta}{\delta j} \ln Z = \frac{\delta W}{\delta j}$$

$$\frac{\delta \Gamma}{\delta \varphi} = -j \implies \Gamma = W - \int j_x \varphi_x$$

Legendre ~~transform~~.

$$\int \frac{-\delta^2 W}{\delta j_x \delta j_z}\frac{\delta^2 \Gamma}{\delta \varphi_z \delta \varphi_y} = \delta_{xy}$$

$$\implies G_{xy} \leftrightarrow \left(\frac{\delta^2 \Gamma}{\delta \varphi_x \delta \varphi_y}\right)^{-1}$$

$$-i\frac{\delta}{\delta j_x} \to \int_z -i\frac{\delta}{\delta j_x}\varphi_z \frac{\delta}{\delta\varphi_z}$$

$$= \int_z -i\frac{\delta^2 W}{\delta j_x \delta j_z}\frac{\delta}{\delta\varphi_z}$$

$$= i\int_z G_{xz}\frac{\delta}{\delta\varphi_z}$$

$$\swarrow$$

$$\Gamma_2^{-1}$$

note

$$\frac{\delta}{\delta\varphi}\Gamma_2^{-1} = \int -\Gamma_2^{-1}\Gamma_3\Gamma_2^{-1}$$

is a suitable generalization.

$$\frac{d}{dx}\frac{1}{f} = \frac{-1}{f}\frac{df}{dx}\frac{1}{f}$$

Z - pic

$$\left(-(\partial^2+m^2)\varphi_x - \frac{\lambda}{6}\varphi_x^3 + j_x\right)\Big|_{\varphi_x \to -i\frac{\delta}{\delta j_x}} Z = 0$$

$$\varphi_x \to -i\frac{\delta}{\delta j_x} + \frac{\delta W}{\delta j_x}$$

W - pic

$$-(\partial^2+m^2)\frac{\delta W}{\delta j_x} - \frac{\lambda}{6}\left(-i\frac{\delta}{\delta j} + \frac{\delta W}{\delta j}\right)^3 \underline{1} + j_x = 0$$

$$j_x \to -\frac{\delta \Gamma}{\delta \varphi_x}$$

$$-i\frac{\delta}{\delta j_x} \to i\int_{x'} G_{xx'}\frac{\delta}{\delta \varphi_{x'}}$$

$$\frac{\delta W}{\delta j_x} \to \varphi_x$$

$$-(\partial^2+m^2)\varphi_x - \frac{\lambda}{6}\left(i\int_{xx'} G_{xx'}\frac{\delta}{\delta\varphi_{x'}} + \varphi_{x'}\right)^3 \underline{1}$$

$$-\frac{\delta\Gamma}{\delta\varphi_x} = 0.$$

Γ - pic

SDE #3

$\frac{d}{d\varphi_y}$ further

$$\frac{d^2\Gamma}{d\varphi_x d\varphi_y} = -(\partial^2 + m^2)\,\delta_{xy}$$

$$-\frac{\lambda}{6}\frac{d}{d\varphi_y}\left\{ i\int G\frac{d}{d\varphi} + \varphi \right\}$$

$$\left\{ i\int G\frac{d}{d\varphi} + \varphi \right\} \varphi.$$

w

$$\frac{d}{d\varphi_z} G_{xy} = -\int_{z',z''} G_{xz'}\,\Gamma_{3\,zz'z''}\,G_{z''y}$$

$$\left\{ i\int G\frac{d}{d\varphi} + \varphi \right\}^3$$

$$\rightarrow \left(i\int G\frac{d}{d\varphi} + \varphi \right)\left(i\,G_{xx} + \varphi_x^2 \right)$$

$$= \int G\,G\,\Gamma_3\,G + 3i\,G_{xx}\,\varphi_x + \dots$$

$\frac{d}{d\varphi_y}$ further.

$$\rightarrow \int G\,G\,\Gamma_4\,G + 3i\,G_{xy}\,\delta_{xy}$$

$$+ \dots$$

↳ vanishing terms

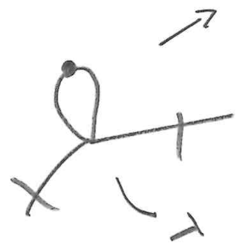
$$\frac{\delta^2 \Gamma}{\delta \varphi_x \, \delta \varphi_y} = \underbrace{-(\partial^2 + m^2)\,\delta_{xy}}_{\sim\, p^2 - m^2} +$$

$$- \frac{\lambda}{2}\, i\, G_{xx}\, \delta_{xy} +$$

$$- \frac{\lambda}{6} \int_{z1,z2,z3} G_{x\,z1}\, G_{x\,z2}\, \Gamma_{4\;z1,z2,z3,y}\, G_{x\,z3}$$

loop: $i G_{xx} = \int \frac{d^4\ell}{(2\pi)^4}\, i G_\ell$

$$\Gamma^{-1} = \Gamma^{-1} -$$



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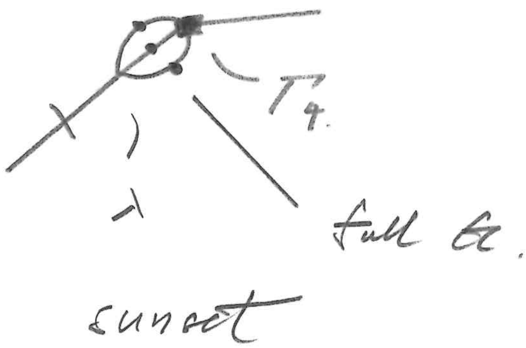

perturbatively, you can show
Gamma4(z1,z2,z3,y) = -lambda delta(z1,y) delta(z2,y) delta(z3,y)