03. 2025

①

Free Scalar Fields

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

Eqn. of Motion (EOM)

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} = \frac{\partial \mathcal{L}}{\delta \varphi}$$

$$(\partial^2 + m^2)\varphi_x = 0$$

$$\pi_x := \frac{\delta \mathcal{L}}{\delta \partial_t \varphi} = \partial_t \varphi_x$$

$$[\varphi_x, \pi_{x'}] = i\,\delta^3_{\vec{x}-\vec{x}'} \quad \text{if} \quad t = t'$$

equal time commutation relation

(ETCR)

Realization #1

$$\varphi_x = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left(e^{-ip\cdot x} a_{\vec{p}} + e^{ip\cdot x} a^{+}_{\vec{p}} \right)$$

$p^0 = \varepsilon_{\vec{p}}$ i.e. on-shell

$$\pi_x = \int \frac{d^3p}{(2\pi)^3} \, -i\sqrt{\frac{\varepsilon_p}{2}} \left(e^{-ip\cdot x} a_{\vec{p}} - e^{ip\cdot x} a^{+}_{\vec{p}} \right).$$

$$\langle 0 | a_{\vec{p}} \, a^{+}_{\vec{p}'} | 0 \rangle = (2\pi)^3 \delta^3_{\vec{p}-\vec{p}'}$$

check that

$$\Rightarrow \quad [\, \varphi_{\vec{x}} , \pi_{\vec{y}} \,] = i \, \delta^3_{\vec{x}-\vec{y}}$$

$$\langle 0 | T\{ \varphi_x \varphi_y \} | 0 \rangle$$

$$= \theta(x^0 - y^0) \langle 0 | \varphi_x \varphi_y | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \varphi_y \varphi_x | 0 \rangle$$

$$= \theta(x^0-y^0) \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_1}} \frac{1}{\sqrt{2\varepsilon_2}} e^{-ip_1\cdot x} e^{ip_2\cdot y} (2\pi)^3 \delta^3_{\vec{p}_1-\vec{p}_2} \quad +$$

$$\theta(y^0-x^0) \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_1}} \frac{1}{\sqrt{2\varepsilon_2}} e^{ip_1\cdot x} e^{-ip_2\cdot y} (2\pi)^3 \delta^3_{\vec{p}_1-\vec{p}_2}$$

the 2 terms summarize nicely by

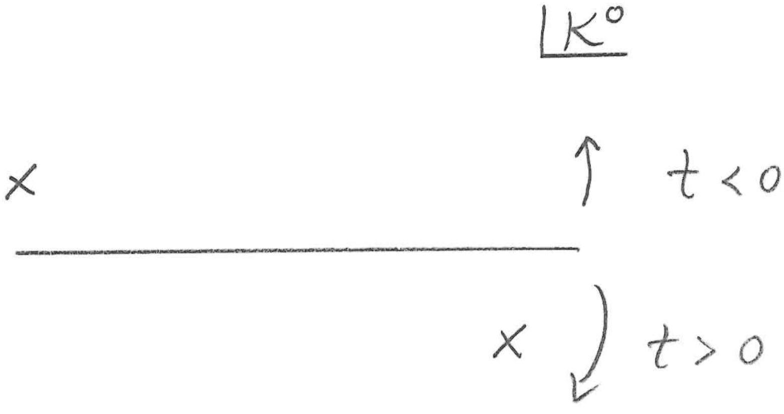
$$\to \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2E_k}\, e^{+i\vec{k}\cdot(\vec{x}-\vec{y}) - iE_{\vec{k}}|x^0-y^0|}$$

↳ check the sign

$$= \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\, e^{-ik\cdot(x-y)}.$$

$\updownarrow$

iG_2



$t = x^0 - y^0$

$$\mathcal{H} = \pi \dot{\varphi} - \mathcal{L}$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2$$

take $t \to 0$ $\quad$ $\mathcal{H}$ is t-independent anyway

$$H = \int d^3x \, \mathcal{H}_x$$

$$\langle 0 | H | 0 \rangle$$

easier to think of $-\varphi \nabla^2 \varphi$

$$= V \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left(\frac{\varepsilon_p}{2} + \frac{\vec{p}^2 + m^2}{2\varepsilon_p} \right)$$

$$= V \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \varepsilon_p \qquad \text{vac. energy}$$

In fact, before sandwiched by $\langle 0 | \cdots | 0 \rangle$

$$\hat{H} = \int d^3x \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} e^{i(\vec{p}_1 + \vec{p}_2)\cdot\vec{x}} \frac{1}{2} \times$$

$$\Big\{ -\sqrt{\frac{\varepsilon_1 \varepsilon_2}{4}} (a_{\vec{p}_1} - a^\dagger_{-\vec{p}_1})(a_{\vec{p}_2} - a^\dagger_{-\vec{p}_2}) +$$

$$\frac{1}{\sqrt{4\varepsilon_1\varepsilon_2}} (-\vec{p}_1\cdot\vec{p}_2 + m^2)(a_{\vec{p}_1} + a^\dagger_{-\vec{p}_1})(a_{\vec{p}_2} + a^\dagger_{-\vec{p}_2}) \Big\}.$$

$$\int d^3x \; e^{i(\vec{p}_1+\vec{p}_2)\cdot\vec{x}} \longrightarrow (2\pi)^3 \delta^3_{\vec{p}_1+\vec{p}_2}.$$

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \, \frac{1}{2} \times \Big\{$$

$$\left(-\frac{\varepsilon_p}{2}\right)\left(a_{\vec{p}}\, a_{-\vec{p}} - a_{\vec{p}}\, a^{+}_{\vec{p}} - a^{+}_{-\vec{p}}\, a_{-\vec{p}} + a^{+}_{-\vec{p}}\, a^{+}_{\vec{p}} \right) +$$

$$\left(\frac{\varepsilon_p}{2}\right)\left(a_{\vec{p}}\, a_{-\vec{p}} + a_{\vec{p}}\, a^{+}_{\vec{p}} + a^{+}_{-\vec{p}}\, a_{-\vec{p}} + a^{+}_{-\vec{p}}\, a^{+}_{\vec{p}} \right) \Big\}$$

$$a^{+}_{\vec{p}}\, a_{\vec{p}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \, \frac{1}{2} \varepsilon_p \left(a_{\vec{p}}\, a^{+}_{\vec{p}} + a^{+}_{-\vec{p}}\, a_{-\vec{p}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \, \varepsilon_p \Big(\underbrace{a^{+}_{\vec{p}}\, a_{\vec{p}}}_{:\hat{H}: \;\text{normal order}} + \frac{1}{2} \underbrace{\left[a_{\vec{p}}, a^{+}_{\vec{p}} \right]}_{(2\pi)^3 \delta^3(\vec{p}-\vec{p})} \Big)$$

$$\langle 0 | :\hat{H}: | 0 \rangle \longrightarrow 0$$

Where is the V factor?

$$(2\pi)^3 \delta^3_{\vec{p}-\vec{p}} \leftrightarrow V.$$

$$\langle 0|\hat{H}|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \varepsilon_p \quad V.$$

as obtained before

Realization # 2.

$$\langle 0 | T\{ \varphi_x \varphi_y \} | 0 \rangle = i G_2$$

$$G_2(x,y) = \int \frac{d^4k}{(2\pi)^3} \frac{1}{k^2 - m^2 + i\delta} e^{-ik\cdot(x-y)}$$

$$(\partial_x^2 + m^2)\, G_2 = -\delta^4_{x-y}$$

or

$$\underline{(\partial_x^2 + m^2) \langle 0 | T\{ \varphi_x \varphi_y \} | 0 \rangle} = -i\,\delta^4_{x-y}.$$

verify LHS directly:

$$(\partial_x^2 + m^2) \left[\theta_{x^0-y^0} \langle \varphi_x \varphi_y \rangle + \theta_{y^0-x^0} \langle \varphi_y \varphi_x \rangle . \right]$$

$$\partial_t^2 \left[\theta(x^0-y^0) \langle \varphi_x \varphi_y \rangle \right]$$

$$= \partial_t \left(\delta_{x^0-y^0} \langle \varphi_x \varphi_y \rangle + \theta(x^0-y^0) \langle \dot\varphi_x \varphi_y \rangle \right)$$

$$= \delta_{x^0-y^0} \langle \dot\varphi_x \varphi_y \rangle + \underline{\theta(x^0-y^0) \langle \ddot\varphi_x \varphi_y \rangle}$$

such term $\to 0$

when $-\nabla^2 + m^2$ acts.

LHS

$$\delta(x^0-y^0) \quad \langle \dot\varphi_x \varphi_y - \varphi_y \dot\varphi_x \rangle$$

$$[\dot\varphi_x, \varphi_y] \rightarrow -i\,\delta^3_{\vec{x}-\vec{y}}\,.$$

at equal time

check the sign

$$(\partial_x^2+m^2)\langle T\{\varphi_x\varphi_y\}\rangle = -i\,\delta^4_{x-y}$$

$$\Rightarrow \quad \langle T\{\varphi_x\varphi_y\}\rangle = \frac{-i}{\partial_x^2+m^2}\,\delta^4_{x-y}\,.$$

$$= \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\delta}\,e^{-ik\cdot(x-y)}$$

//

Realization 3.

wait till
Functional approach ...

Dirac Fields

03. 2025

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m\mathbb{I})\psi$$

Eqn. of Motion

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} = \frac{\delta \mathcal{L}}{\delta \psi} \qquad \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\psi}} = \frac{\delta \mathcal{L}}{\delta \bar{\psi}}$$

$$\Rightarrow \partial_\mu \bar{\psi} i \gamma^\mu = -\bar{\psi} m \qquad 0 = (i\not{\partial} - m)\psi$$

$$\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0 \quad \longleftrightarrow \quad (i\not{\partial} - m)\psi = 0$$

no new info.

$$\bar{\psi} = \psi^\dagger \gamma^0$$

$$[(i\not{\partial} - m)\psi]^\dagger = 0$$

$$(-i\partial_\mu \psi^\dagger \gamma^{\mu\dagger}) - \psi^\dagger m = 0$$

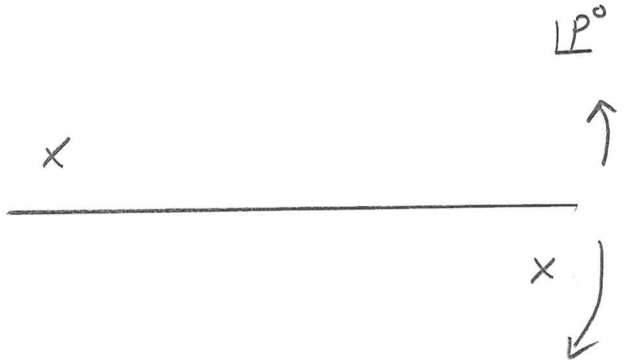
$$\gamma^{\mu\dagger} = \begin{cases} \gamma^0 & \text{if } \mu = 0 \\ -\gamma^i & \text{if } \mu = i \end{cases}$$

$$\rightarrow \gamma^0 \gamma^\mu \gamma^0.$$

$$-(i\bar{\psi}\overleftarrow{\not{\partial}} + \bar{\psi} m)\gamma^0 = 0$$

$$iS_F = \langle 0| T\{\psi_x \bar{\psi}_y\} |0\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4}\, i\, \frac{\not{p}+m}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)}.$$



$x^0-y^0<0$

$x^0-y^0>0$

(extra -1)

$$= \begin{cases} \int \frac{d^3p}{(2\pi)^3} \frac{\not{p}+m}{2E_p} e^{-ip\cdot(x-y)} \quad p^0 = E_p. & x^0-y^0>0 \\ \int \frac{d^3p}{(2\pi)^3} \frac{\tilde{\not{p}}+m}{2E_p} e^{-i\tilde{p}\cdot(x-y)} & x^0-y^0<0. \end{cases}$$

$$\tilde{p}^0 = -E_p$$

$$\Updownarrow$$

$$\int \frac{d^3p}{(2\pi)^3}\left[-\frac{\not{p}-m}{2E_p}\right] e^{ip\cdot(x-y)}$$

Canonical quantization.

$$\psi_{t,\vec{x}} = \sum_s \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{m}{\mathcal{E}_p}} \left[u_{\vec{p}s}\, e^{-ip\cdot x}\, b_{\vec{p}s} + v_{\vec{p}s}\, e^{ip\cdot x}\, d^{+}_{\vec{p}s} \right]$$

$$u_{\vec{p}s} = \sqrt{\frac{\mathcal{E}_p + M}{2m}} \begin{bmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{\mathcal{E}_p + m}\chi_s \end{bmatrix}$$

$$v_{\vec{p}s} = \sqrt{\frac{\mathcal{E}_p + M}{2m}} \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{\mathcal{E}_p + m}\chi_s^c \\ \chi_s^c \end{bmatrix}$$

$$\chi_s^c = -i\sigma_2 \chi_s^*$$

$$\{ b_{\vec{p}s}, b^{+}_{\vec{p}'s'} \} = (2\pi)^3 \delta^3_{\vec{p}-\vec{p}'} \delta_{ss'}$$

similarly for d

$$\{ b, d \} = 0$$

$s \rightarrow$ 2 labels $(\pm)$ for spin $1/2$

e.g. $\chi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\chi_s^c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

useful facts about spinors

$u_{\vec{p}s}$ has 4 components

$\overline{u}_{\vec{p}s_1} u_{\vec{p}s_2} \longleftrightarrow$

$$\sum_{\alpha=1,2,3,4} \overline{u}^{\alpha}_{\vec{p}s_1} u^{\alpha}_{\vec{p}s_2}$$

$$= \frac{\varepsilon_p + m}{2m} \left(1 - \frac{\vec{p}^{\,2}}{(\varepsilon_p+m)^2} \right) \chi^{+}_{s_1} \chi_{s_2}$$

$$= \frac{\varepsilon_p + m}{2m} \left(1 - \frac{\varepsilon_p - m}{\varepsilon_p + m} \right) \delta_{s_1 s_2}$$

$$= \delta_{s_1 s_2}$$

but

$$\overline{v}_{\vec{p}s_1} v_{\vec{p}s_2} = \frac{\varepsilon_p + m}{2m} \left(\frac{\vec{p}^{\,2}}{(\varepsilon_p+m)^2} - 1 \right) \chi^{c+}_{s_1} \chi^{c}_{s_2}$$

$$= -\delta_{s_1 s_2} \qquad \text{extra } -1$$

even more combination

$$\sum_s u_{\vec{p}s} \bar{u}_{\vec{p}s} = \frac{\varepsilon_p + m}{2m} \sum_s \begin{bmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{\varepsilon_p+m}\chi_s \end{bmatrix} \begin{bmatrix} \chi_s^+ & -\chi_s^+ \frac{\vec{\sigma}\cdot\vec{p}}{\varepsilon_p+m} \end{bmatrix}$$

$$= \frac{1}{2m} \begin{bmatrix} (\varepsilon_p+m)\mathbb{I}_2 & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & \frac{-\vec{p}^2}{\varepsilon_p+m}\mathbb{I}_2 \end{bmatrix}$$

note

$$\sum_s \chi_s \chi_s^+ = \mathbb{I}_2$$

$$\frac{-\vec{p}^2}{\varepsilon_p+m}\mathbb{I}_2 \leftrightarrow (-\varepsilon_p + m)\mathbb{I}_2$$

$$= \frac{1}{2m}\left(\varepsilon_p \gamma^0 - \vec{p}\cdot\vec{\gamma} + m\mathbb{I}_4\right).$$

$$= \frac{\not{p} + m}{2m}$$

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & \\ & -\mathbb{I}_2 \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} & \vec{\sigma} \\ -\vec{\sigma} & \end{pmatrix}$$

Similarly.

$$\sum_s v_{\vec{p}s} \bar{v}_{\vec{p}s} = \frac{\not{p} - m}{2m}$$

check that

$$\mathrm{tr}\left(\sum_s u\bar{u}\right) = \mathrm{tr}\,\mathbb{1}_4 \frac{1}{2} = 2$$

recall

$$\bar{u}_{s_1} u_{s_2} = \delta_{s_1 s_2}$$

$$\sum_s \delta_{ss} = 2$$

$$\mathrm{tr}\left(\sum_s v\bar{v}\right) = -2 = \sum_s (\bar{v}_s v_s).$$

More magic :

$$u^\dagger_{\vec{p}s_1} u_{\vec{p}s_2} = \frac{\varepsilon_p + m}{2m}\left[1 + \frac{\vec{p}^{\,2}}{(\varepsilon_p+m)^2}\right]\delta_{s_1 s_2}$$

$$= \frac{\varepsilon_p}{m}\delta_{s_1 s_2}$$

$$v^\dagger_{\vec{p}s_1} v_{\vec{p}s_2} = \frac{\varepsilon_p + m}{2m}\left(\frac{\vec{p}^{\,2}}{(\varepsilon_p+m)^2} + 1\right)\delta_{s_1 s_2}$$

$$= \frac{\varepsilon_p}{m}\delta_{s_1 s_2}$$

$\longrightarrow$ Non-rel physics : $\bar{u}u \approx u^\dagger u \approx v^\dagger v$

but $\bar{v}v \to -1$ extra factor.

these can be obtained via

$$\mathrm{tr} \sum_s u_s u_s^\dagger = \mathrm{tr}\left(\frac{\not{p}+m}{2m}\gamma^0\right)$$

$$= 2\,\frac{\mathcal{E}_p}{m}$$

$$\mathrm{tr} \sum_s v_s v_s^\dagger = \mathrm{tr}\left(\frac{\not{p}-m}{2m}\gamma^0\right).$$

$$= 2\,\frac{\mathcal{E}_p}{m}$$

last, but not least

$$\bar{u}_{p s_1}\, v_{\vec{p} s_2} = \frac{\mathcal{E}_p+m}{2m}\left[\chi_{s_1}^\dagger \quad \chi_{s_1}^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{\mathcal{E}_p+m}\right] \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{\mathcal{E}_p+m}\chi_{s_2}^c \\ -\chi_{s_2}^c \end{bmatrix}$$

$$= 0 \quad //$$

but

$$u_{p s_1}^\dagger\, v_{\vec{p} s_2} = \frac{\mathcal{E}_p+m}{2m}\; 2\, \chi_{s_1}^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{\mathcal{E}_p+m}\chi_{s_2}^c$$

$$= \frac{\vec{p}}{m}\cdot\left(\chi_{s_1}^\dagger\, \vec{\sigma}\, \chi_{s_2}^c\right)$$

Non-zero

momentum can talk to the spin ...

Propagator

$$iS_F = \langle 0 | T\{ \psi_x \bar{\psi}_y \} | 0 \rangle.$$

$$= \theta(x^0 - y^0) \langle 0 | \psi_x \bar{\psi}_y | 0 \rangle$$

$$+ \theta(y^0 - x^0) \langle 0 | (- \bar{\psi}_y \psi_x) | 0 \rangle$$

no sum

$$\langle 0 | \psi_x \bar{\psi}_y | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{m}{\mathcal{E}_p} u_s \bar{u}_s \, e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\not{p} + m}{2\mathcal{E}_p} e^{-ip\cdot(x-y)}$$

$$p^0 = \mathcal{E}_p$$

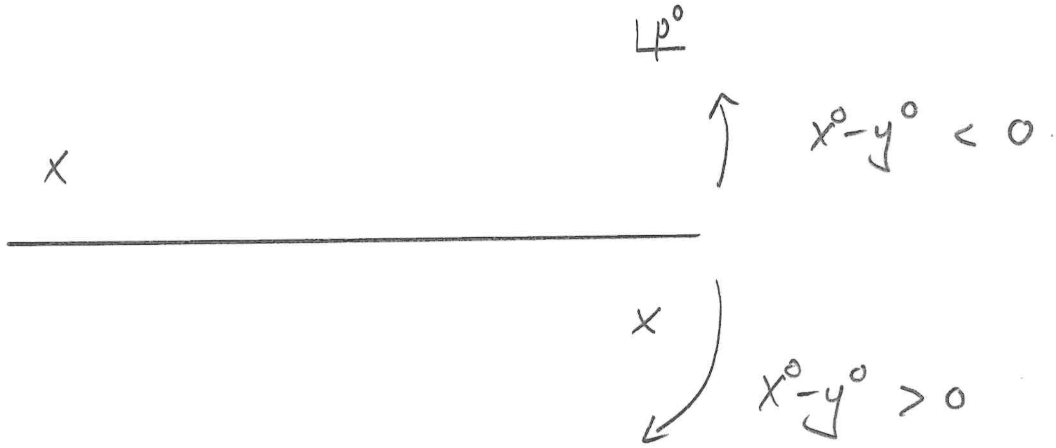
$$-\langle 0 | \bar{\psi}_y \psi_x | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{-m}{\mathcal{E}_p} v_s \bar{v}_s \, e^{ip\cdot(x-y)}.$$

$$= \int \frac{d^3p}{(2\pi)^3} \left[-\left(\frac{\not{p} - m}{2\mathcal{E}_p} \right) \right] e^{ip\cdot(x-y)}$$

$$p^0 = \mathcal{E}_p.$$

$$\frac{i}{\not p - m}$$

$$iS_F = \int \frac{d^4p}{(2\pi)^4}\, i\,\frac{\not p + m}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x-y)}.$$



when $x^0 - y^0 < 0$

$$iS_F \to \int \frac{d^3p}{(2\pi)^3}\, \frac{\tilde{\not p} + m}{2E_p}\, e^{-i\tilde p\cdot(x-y)}$$

$\tilde p^0 = -E_p$

$$\int \frac{d^3p}{(2\pi)^3}\left[-\frac{\not p - m}{2E_p}\right] e^{+ip\cdot(x-y)}$$

after flipping $\vec p$

$$\mathcal{H} = \pi_\psi \dot\psi + \pi_{\bar\psi} \dot{\bar\psi} - \mathcal{L}$$

$$\pi_\psi = \frac{\delta \mathcal{L}}{\delta \partial_t \psi} = i\bar\psi\gamma^0 = i\psi^\dagger \qquad \pi_{\bar\psi} = \frac{\delta \mathcal{L}}{\delta \partial_t \bar\psi} = 0.$$

$$\mathcal{H} \to \bar\psi(-i\vec\gamma\cdot\nabla + m)\psi$$

$$\updownarrow$$

$$\hat h = \vec\gamma\cdot\vec p + m\,\mathbb{I}_4.$$

$$\vec L = \vec x \times \vec p$$

$$[\varepsilon^{ijk} x^j p^k,\ \gamma^\ell p^\ell]$$

care less abt the $\vec\gamma$ structure

$$= \varepsilon^{ijk} p^k \gamma^\ell (i\delta_{j\ell})$$

$$= i\,\vec\gamma\times\vec p$$

$$\vec S = \frac{1}{2}\begin{bmatrix}\vec\sigma & \\ & \vec\sigma\end{bmatrix} \qquad \vec\gamma = \begin{pmatrix} & \vec\sigma \\ -\vec\sigma & \end{pmatrix}.$$

$$[S^i,\ \vec\gamma\cdot\vec p] = \begin{bmatrix} 0 & \frac{1}{2}(\sigma^i\sigma^j - \sigma^j\sigma^i)p^j \\ -\frac{1}{2}[\sigma^i,\sigma^j]\,p^j & 0 \end{bmatrix}$$

$$= \begin{bmatrix} & i\varepsilon^{ijk}\sigma^k p^j \\ -i\varepsilon^{ijk}\sigma^k p^j & \end{bmatrix} = -i\,\vec\gamma\times\vec p$$

$$[\underline{\vec{L}+\vec{S}}, \hat{h}] = \vec{0}$$

$\vec{J} = \vec{L}+\vec{S}$ but $[\vec{L}, \hat{h}] \neq 0$

$[\vec{S}, \hat{h}] \neq 0$

l, s are no longer good quantum no.

$$\mathcal{E}_{vac} = \frac{E_{vac}}{V} = \langle 0 | \mathcal{H} | 0 \rangle$$

$$= \langle 0 | \overline{\bar{\psi}_x (-i\vec{\gamma}\cdot\nabla + m) \psi_x} | 0 \rangle$$

$$= tr\left[(-i\vec{\gamma}\cdot\nabla + m)\, -iS_{xx'} \right]$$

$x' \to x$

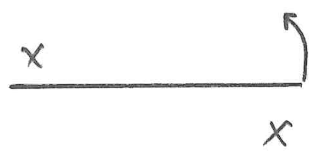
w $x'^0 > x^0$

i.e. $t \to 0^-$

$$iS_{xx'} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2 - m^2 + i\delta} e^{-ip\cdot(x-x')}$$

$x' \to x$

$x^0 < x'^0$



$$\longrightarrow \int \frac{d^3p}{(2\pi)^3} \frac{\tilde{\not{p}}+m}{2\mathcal{E}_p} e^{-i\tilde{p}\cdot(x-x')}$$

$\tilde{p}^0 = -\mathcal{E}_p$.

$x' \to x$

not yet! (derivative)

$$\mathcal{E}_{vac} \to -\int \frac{d^3p}{(2\pi)^3}\, tr\left[(\vec{p}\cdot\vec{\gamma} + m)\, \frac{-\mathcal{E}_p \gamma^0 - \vec{p}\cdot\vec{\gamma} + m}{2\mathcal{E}_p} \right]$$

$$= -\int \frac{d^3p}{(2\pi)^3}\, 4\, \frac{\vec{p}^2 + m^2}{2\mathcal{E}_p}.$$

$$= 4 \times \quad -\int \frac{d^3p}{(2\pi)^3} \frac{1}{2}\mathcal{E}_p.$$

$p\ \bar{p}$
spin $1/2$

negative of
scalar.

$$dE = -p\,dV + T\,dS + \mu\,dN$$

vac : $T \to 0 \quad (dN = 0)$

$$P_{vac} = -\left(\frac{\partial E}{\partial V}\right)_{S,N}$$

$$= -\mathcal{E}_{vac}.$$

or $\quad \ln Z = -V_4 f \simeq -V_4\, \mathcal{E} = V_4\, P$

$$V_4 = \beta V$$

$\beta = 1/T \to \infty \quad \sim$ long Euclidean time even cooler way:

$$\det(\not{p} - m)$$

$$= \det\begin{pmatrix} E-m & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & -E-m. \end{pmatrix}$$

$$= \det\begin{bmatrix} E-m & 0 & p_z & p_x+ip_y \\ 0 & E-m & p_x-ip_y & -p_z \\ -p_z & -p_x-ip_y & -E-m & 0 \\ -p_x+ip_y & p_z & 0 & -E-m. \end{bmatrix}$$

$$= (E-m)\left[(E-m)(E+m)^2 - p_z^2(E+m) - (p_x^2+p_y^2)(E+m)\right]$$

$$- 0 +$$

$$p_z\left[p_z^3 + p_z(p_x^2+p_y^2) - p_z(E^2-m^2)\right]$$

$$- (p_x+ip_y)\left[(E^2-m^2)(p_x-ip_y) - p_z^2(p_x-ip_y) - (p_x^2+p_y^2)(p_x-ip_y)\right]$$

$$= (E^2 - \vec{p}^2 - m^2)^2$$

brute force

OMG.

$\det \not{p} - m$

$$= \det(\gamma_5(\not{p}-m)\gamma_5)$$

$$= \det(-\not{p}-m)$$

$$= \sqrt{\det(\not{p}-m)\ \det(-\not{p}-m)}$$

$$= \sqrt{\det[(-p^2+m^2)\,\mathbb{I}_4]}$$

$$= \sqrt{(p^2-m^2)^4} \quad = (p^2-m^2)^2$$

$\det(ABC) =$

$\det A \det B \det C$

$= \det AC \ \det B$

$\gamma_5^2 = I$

degen $\longrightarrow$ power.

$\sqrt{\ } \longrightarrow$ cut

Now

$$\ln Z = \mathrm{tr} \ln(i\not{\partial}-m)$$

$$= \sum_p \mathrm{tr}_D \ln \not{p}-m$$

$$= \sum_p \frac{1}{2}\, 4 \ \ln(p^2-m^2)$$

using the above ...

$$\sum_p \ln \det \not{p}-m$$

$$= \sum_p 2\ln(p^2-m^2)$$

$$= V_4 \int \frac{dp_4}{2\pi}\frac{d\vec{p}}{(2\pi)^3}\ \frac{1}{2}(4)\ \ln(p_4^2+\vec{p}^{\,2}+m^2)$$

$$\int \frac{dp_4}{2\pi} \ln(p_4^2+\varepsilon_p^2) = \varepsilon_p$$

by e.g. Schwinger's eg.

$$\ln Z_{\text{vac.}} = \mathrm{tr} \ln (i\partial\!\!\!/ - m)$$

$$= V_4 \int \frac{d^4p}{(2\pi)^4} \, \mathrm{tr}_D \ln (p\!\!\!/ - m).$$

$$= V_4 \int \frac{d^3p}{(2\pi)^3} \, \frac{1}{2} \, 4 \, \varepsilon_p.$$

$$= - V_4 \, \varepsilon_{vac}.$$

Gauge Fields

$$\mathcal{L} = -\frac{1}{4} F^2 - j_\mu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} = \frac{\delta \mathcal{L}}{\delta A_\nu}$$

4 terms $\partial_\mu A_\nu$
each gives $F^{\mu\nu}$

$$-\partial_\mu F^{\mu\nu} = -j^\nu$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

Eqn. of Motion
a.k.a. Maxwell's eqn

$\Rightarrow$ in terms of $\vec{E}$ & $\vec{B}$

$$E^i = F^{i0} = \partial^i A^0 - \partial^0 A^i$$

$$\vec{E} = -\nabla A^0 - \partial_t \vec{A}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$B^k = \varepsilon^{kij'} \partial_{i'} A^{j'} = -\frac{1}{2} \varepsilon^{kij'} F^{ij'}$$

$$\varepsilon^{kij} B^k = \partial_i A^j - \partial_j A^i$$

$$= -F^{ij}$$

The Maxwell's eqns:

start from

$$\partial_\mu F^{\mu\nu} = j^\nu$$

Gauss law.

$\nu = 0$ $\quad \partial_i F^{i0} = j^0 \Rightarrow \nabla \cdot \vec{E} = \rho$

$\nu = i$

Ampere's law.

$$\partial_t F^{0i} + \partial_j F^{ji} = j^i \Rightarrow \underline{-\partial_t \vec{E}} + \nabla \times \vec{B} = \vec{J}$$

correction from Maxwell.

$$\nabla \times \vec{B} = \vec{J} + \partial_t \vec{E}$$

how to get the other 2 ?

$\Rightarrow$ based on def. of $\vec{E}, \vec{B}$ from $\vec{A}$

We need $\nabla \cdot \vec{B}$ & $\nabla \times \vec{E}$ to finish

$$\vec{B} = \nabla \times \vec{A}$$

this dictates $\nabla \cdot \vec{B} = 0$

$$\vec{E} = -\nabla A^0 - \partial_t \vec{A}$$

$$\nabla \times \vec{E} = -\partial_t \nabla \times \vec{A} = -\partial_t \vec{B}$$

Faraday's law.

Fancier way to see this

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

$$\Rightarrow \partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0$$

Bianchi Identity

e.g. $\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$

$\Rightarrow \partial_j B_j = 0 .$

Also current conservation is for free

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\underbrace{\partial_\nu \partial_\mu}_{\text{sym}} \underbrace{F^{\mu\nu}}_{\text{anti-sym}} = 0 = \partial_\nu j^\nu .$$

$$\partial_t \rho + \nabla \cdot \vec{J} = 0$$

$$F^{\mu\nu} \longleftrightarrow \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\mu\nu} \longleftrightarrow \begin{matrix} \vec{E} \to -\vec{E} \\ \vec{B} \to \vec{B} \end{matrix} \quad \text{of } F^{\mu\nu}$$

$$\mathcal{L}_{vac.} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$= -\frac{1}{4} \left\{ \underbrace{2 F_{0i} F^{0i}}_{-2\vec{E}^2} + \underbrace{F_{ij} F^{ij}}_{2\vec{B}^2} \right\}$$

$$= \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right)$$

$$\mathcal{H}_{vac} = \pi^i \underline{(\partial_t A_i)} - \mathcal{L}_{vac}$$

$$\hookrightarrow F_{0i} + \partial_i A_0 \longleftrightarrow \vec{E} + \nabla A^0$$

$$\frac{\delta \mathcal{L}}{\delta \partial_t A_i} = -F^{0i} = E^i$$

ugly, but $\longrightarrow 0$ by $\nabla \cdot \vec{E} = 0$

$$\longrightarrow \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) + \vec{E} \cdot \nabla A^0$$

KE

like a pot. term

Gauge Symmetry

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

for any $\alpha(x)$ $\longrightarrow$ the same $F_{\mu\nu}$

$\longrightarrow$ same $\vec{E}$, $\vec{B}$

$\longrightarrow$ same physics

freedom to choose $\alpha(x)$

$\updownarrow$

freedom to choose a
gauge fixing condition
for $A_\mu(x)$

e.g.

$\partial_\mu A^\mu = 0$ Lorenz gauge
not Lorentz

$\nabla \cdot \vec{A} = 0$ Coulomb

$A^0 = 0$ temporal

$A^3 = 0$ axial

Most gauge fixing schemes are <u>NOT unique</u> !

e.g. $\partial_\mu A^\mu = 0$

$A' = A + \partial\alpha'$

$\partial_\mu A'^\mu = 0$ as long as $\partial^2 \alpha' = 0$

if we pick $\partial_t \alpha' = -A_0$ further

$\Rightarrow A'_0 = 0$ — temporal.

$\Rightarrow \partial_\mu A'^\mu = 0 \Rightarrow \nabla \cdot \vec{A}' = 0$ Coulomb gauge

$\vec{A}(x)$ has only 2 DoFs !

(physical gauge)

E.O.M

$$\partial_\mu F^{\mu\nu} = 0 = \partial^2 A^\nu - \underbrace{\partial^\nu \partial_\mu A^\mu}_{0}$$

$\partial^2 A^\nu = 0$ $A^0 = 0$ by choice.

$\nabla \cdot \vec{A} = 0 \Rightarrow \vec{A}$ is transverse !

Canonical Quantization

$$\pi^\mu = \frac{\delta \mathcal{L}}{\delta \partial_t A_\mu} \longrightarrow \pi^0 = 0$$

there is no $\partial_t A_0$ term in $\mathcal{L}$.

$$\longrightarrow \pi^i = -F^{0i} = E^i$$

$$[A_{i\,\vec{x}}, \pi^j_{\vec{y}}] \stackrel{?}{=} i\delta_{ij}\,\delta^3_{\vec{x}-\vec{y}}$$

(π^i conjugate to $\underline{A_i} = -A^i$)

if

$$[A^i_{\vec{x}}, E^j_{\vec{y}}] \stackrel{?}{=} -i\,\delta_{ij}\,\delta^3_{\vec{x}-\vec{y}}$$

$$\partial_{i\,x}[A^i, \pi^j] = [\nabla\cdot\vec{A}, \pi^j] = \underline{i\,\partial^j_x\,\delta^3_{\vec{x}-\vec{y}}}$$

should be 0 $\longrightarrow$ Non-zero.

but it's not

$\Rightarrow$

$$[A^i_{\vec{x}}, E^j_{\vec{y}}] = -i\,\underline{\Delta^{ij}}\,\delta^3_{\vec{x}-\vec{y}}$$

$$\hookrightarrow \left(\delta^{ij} - \frac{\nabla_i\nabla_j}{\nabla^2}\right)$$

$$= -i\int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(\delta^{ij} - \frac{k^ik^j}{k^2}\right)$$

check that $\partial_i[A^i, E^j] = 0$ //

$$A^\mu_x = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=1,2} \left[\varepsilon^\mu_\lambda a_{\vec{p}\lambda} e^{-ip\cdot x} + \varepsilon^{*\mu}_\lambda a^\dagger_{\vec{p}\lambda} e^{ip\cdot x} \right]$$

ε^μ: 4 degree of freedom

C-gauge:

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \quad p_\mu \varepsilon^\mu = 0$$

$$A^0 = 0 \quad \Rightarrow \quad \varepsilon^0 = 0$$

$$\vec{p} \cdot \vec{\varepsilon} = 0$$

$$\vec{\varepsilon} \rightarrow 2 \text{ DOFs.}$$

Polarization tensor : $\varepsilon^\mu_{\lambda=1,2}$ e.g. $\vec{p} \parallel \hat{z}$

$$\varepsilon^\mu_{\lambda=1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \varepsilon^\mu_{\lambda=2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{\varepsilon^{*\mu}_\lambda \varepsilon_{\lambda'\mu}} = -\delta_{\lambda\lambda'} \rightarrow g_{\lambda\lambda'}$$

(if we pick the label for λ right)

$$\swarrow \qquad \qquad \Leftarrow$$

$$g_{\mu\nu} \varepsilon^{*\mu}_\lambda \varepsilon^\nu_{\lambda'}$$

$$\sum_\lambda \varepsilon^\mu_\lambda \varepsilon^{*\nu}_\lambda = \begin{bmatrix} 0 & \\ & \left[\delta^{ij} - \frac{p^i p^j}{\vec{p}^2}\right] \end{bmatrix}$$

ε^μ helps to realize the gauge conditions :

$$\nabla\cdot\vec{A} = 0 \ ; \ A^0 = 0$$

$$\vec{A} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\varepsilon_{\vec{p}}}}\sum_{\lambda=1,2}\left[\vec{\varepsilon}_\lambda\, a_{\vec{p}\lambda}\, e^{-ip\cdot x} + \vec{\varepsilon}_\lambda^{\,*}\, a^\dagger_{\vec{p}\lambda}\, e^{ip\cdot x}\right]$$

$$\varepsilon_{\vec{p}} = |\vec{p}|$$

$$\vec{E} = -\cancel{\nabla A^0} - \partial_t\vec{A}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, i\sqrt{\frac{\varepsilon_{\vec{p}}}{2}}\sum_{\lambda=1,2}\left[\vec{\varepsilon}_\lambda\, a_{\vec{p}\lambda}\, e^{-ip\cdot x} - \vec{\varepsilon}_\lambda^{\,*}\, a^\dagger_{\vec{p}\lambda}\, e^{ip\cdot x}\right]$$

$$\left[-\vec{A}_{\vec{x}}, \vec{E}_{\vec{y}}\right] = i\,\delta^3_{\vec{x}-\vec{y}}$$

& no issues in handling $\nabla\cdot\vec{A}$

$$\mathcal{H} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$$

$$= \frac{1}{2}\left[\dot{\vec{A}}^2 + \underline{(\nabla\times\vec{A})^2}\right]$$

$$= \varepsilon_{ijk}\,\varepsilon_{ij'k'}\left[\partial_j A^k\, \partial_{j'} A^{k'}\right]$$

$$\rightarrow -\vec{A}\,\nabla^2\vec{A} \qquad \text{as } \nabla\cdot\vec{A} = 0$$

$$= \frac{1}{2}\vec{A}\cdot\left(-\partial_t^2 - \nabla^2\right)\vec{A}$$

both yield the same result ...

$$\leftrightarrow \quad \vec{A}\cdot\left(-\nabla^2\right)\vec{A}$$

$$\hat{H} = \int d^3x \, \mathcal{H}$$

$$\to \int d^3x \; A^i - \nabla^2 A^i$$

flip $\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda=1,2} \frac{1}{2}\varepsilon_{\vec{p}} \left(a^\dagger_{\vec{p}\lambda} a_{\vec{p}\lambda} + a^\dagger_{-\vec{p}\lambda} a_{-\vec{p}\lambda} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda=1,2} \varepsilon_{\vec{p}} \left\{ a^\dagger_{\vec{p}\lambda} a_{\vec{p}\lambda} + \frac{1}{2}\left[a_{\vec{p}\lambda} , a^\dagger_{\vec{p}\lambda} \right] \right\}$$

$$:\hat{H}: \qquad\qquad (2\pi)^3 \delta^3_{\vec{p}-\vec{p}} \;\leftrightarrow\; V$$

$$= :\hat{H}: + E_{vac}.$$

w. $$E_{vac} \to \underset{\substack{\uparrow \\ \text{2 polarizations}}}{2} \; \underbrace{V \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \varepsilon_{\vec{p}}}_{\text{1 boson result}}$$

$$\langle 0 | T\{ A^i_x \, A^j_y \} | 0 \rangle$$

$$\sum_\lambda \varepsilon^i_\lambda \varepsilon^{*j}_\lambda$$

$$\updownarrow$$

$$\to \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot(x-y)} \, \frac{i}{p^2 + i\epsilon} \left(\delta^{ij} - \frac{p^i p^j}{\vec{p}^{\,2}} \right).$$

there are only 2 propagator, as seen from

$$\mathrm{tr}\left(\delta^{ij} - \frac{p^i p^j}{\vec{p}^{\,2}} \right) = 3 - 1 = 2.$$

We have them all, at the expense of killing Lorentz covariance ...

Some say it is bad, but I don't care

to prepare for an alternative formulation respecting Lorentz covariance:

$$\eta^\mu := [1, 0, 0, 0] \quad \text{timelike}$$

it means $\hat{k}^0_\parallel = 0$

$$\hat{k}^\mu_\parallel := \frac{k^\mu - (k\cdot\eta)\,\eta^\mu}{\sqrt{(k\cdot\eta)^2 - k^2}} \quad \text{s.t.} \quad \hat{k}_\parallel \cdot \eta = 0$$

$$\hat{k}_\parallel \cdot \hat{k}_\parallel = -1$$

$$\hat{k}_\parallel \cdot \varepsilon \propto \vec{k} \cdot \vec{\varepsilon} \to 0$$

$$\eta \cdot \varepsilon = 0 \quad (\text{by def})$$

$$\sum_\lambda \varepsilon^\mu_\lambda \varepsilon^{*\nu}_\lambda \to$$

$$-\left(g^{\mu\nu} - \eta^\mu \eta^\nu + \hat{k}^\mu_\parallel \hat{k}^\nu_\parallel \right)$$

the propagator can be written as

$$i D_{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2+i\delta} e^{-ip\cdot x} \times$$

$$\Big\{ \underbrace{-g_{\mu\nu}}_{\substack{\text{"magically" gives} \\ 0 \text{ if } \mu,\nu \to 0 \\ \text{\& transverse}}} + \underbrace{\frac{-k^2\eta^\mu\eta^\nu - k^\mu k^\nu + (k\cdot\eta)^2(\eta^\mu k^\nu + \eta^\nu k^\mu)}{(k\cdot\eta)^2 - k^2}}_{\text{ugly mess}} \Big\}$$

Covariant gauges

$$[A_\mu, \pi_\nu] = i g_{\mu\nu} \delta^3_{\vec{x}-\vec{y}}$$ at equal time.

not possible as $\pi_0 = 0$

$\longrightarrow$ need to modify $\mathcal{L}$

by adding a mass term

$$-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

$$\frac{\delta \mathcal{L}}{\delta \partial_t A_\mu} = -F^{0\mu} - \frac{1}{\xi} g^{\mu 0}(\partial \cdot A)$$

π^i remains the same : $\vec{E}$

$$\pi^0 = -\frac{1}{\xi}(\partial_\mu A^\mu) \neq 0$$

EOM

$$-\partial_\mu F^{\mu\nu} - \frac{1}{\xi}\partial_\mu g^{\mu\nu}(\partial \cdot A) = 0$$

$$\partial^2 A^\nu - \underline{(1-\xi^{-1})}\,\partial^\nu(\partial_\mu A^\mu) = 0$$

e.g. the choice for $\xi = 1$: Feynman gauge.

we recover the original theory

if $\partial_\mu A^\mu = 0$

but π^0 depends on it

→ can't do it in Lagrangian level!

⇒ need to restrict it

to states / expectation values

$$|\alpha\rangle_{phy} \quad = \quad {}_{phy}\langle\alpha|\, \partial\cdot A\, |\alpha\rangle_{phy} = 0.$$

$$A_\mu = \int \frac{d\vec{p}^3}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_\lambda \Big[\varepsilon_\mu^\lambda a_{\vec{p}\lambda} e^{-ip\cdot x} + \varepsilon_\mu^{*\lambda} a_{\vec{p}\lambda}^\dagger e^{ip\cdot x} \Big]$$

need $\forall\ \lambda = 0, 1, 2, 3.$

$$\varepsilon^\lambda \cdot \varepsilon^{\lambda'} = g^{\lambda\lambda'}$$

$$\sum_\lambda \varepsilon_\lambda^\mu \varepsilon_\lambda^\nu \to -\left[g^{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right]$$

↕ inverse of $\left(g_{\mu\nu} - (1-\xi^{-1}) \frac{p_\mu p_\nu}{p^2} \right)$

$$\left[a_{\vec{p}\lambda}, a_{\vec{p}'\lambda'}^\dagger \right] = -(2\pi)^3 \delta^3_{\vec{p}-\vec{p}'}\, g_{\lambda\lambda'}$$

but $\lambda = 0$ $\quad [a_{\vec{p}0}, a_{\vec{p}0}^\dagger] = -(2\pi)^3 \delta^3_{\vec{p}-\vec{p}'}$

↳ ghost state

$$:\hat{H}: \; \propto \; \sum_{\lambda=1}^{3} a_\lambda^\dagger a_\lambda - a_0^\dagger a_0$$

we need to impose on $|\psi\rangle_{phy}$.

$$(a_3 - a_0)\,|\psi\rangle_{phy} = 0$$

$${}_{phy}\langle \; :\hat{H}: \; \rangle_{phy} \;\leftrightarrow\; {}_{phy}\langle \int_{\vec{p}} \sum_{\lambda=1,2} \varepsilon_p \, a_{\vec{p}\lambda}^\dagger a_{\vec{p}\lambda} \rangle_{phy}$$

almost cheating
may as well
do it by hand...

Massive Vector Fields

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$$

worry about the sign?

$\vec{A}$ is physical.

$\longrightarrow -\frac{1}{2}m^2 \vec{A}\cdot\vec{A}$ as needed

Equ. of motion.

$$-\partial_\mu F^{\mu\nu} = m^2 A^\nu$$

∂_ν further

$$\Rightarrow \partial_\nu A^\nu = 0$$

No gauge freedom! Has to be this!

$$(\partial^2 + m^2)A^\nu = 0$$

$$\pi^\mu = \frac{\delta \mathcal{L}}{\delta \partial_t A_\mu} = -F^{0\mu}$$

$\pi^0 = 0$ again! But No worry

$\partial_\mu A^\mu = 0 \Rightarrow A^0$ can be eliminated

$$\partial_t A^0 = \nabla\cdot\vec{A}$$

3 physical DoF's.

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\lambda=1,2,3}\varepsilon^\mu_\lambda\left[a_{\vec{p}\lambda}e^{-ip\cdot x} + a^\dagger_{\vec{p}\lambda}e^{ip\cdot x}\right]$$

Normal staffs.

$$\varepsilon_\lambda \cdot \varepsilon_{\lambda'} = g_{\lambda\lambda'}$$

$$[a_{\vec{p}\lambda}, a^\dagger_{\vec{p}'\lambda'}] = \delta_{\lambda\lambda'} (2\pi)^3 \delta^3_{\vec{p}-\vec{p}'}$$

also $\partial_\mu A^\mu = 0$

$$\Rightarrow \quad p_\mu \varepsilon^\mu = 0$$

$$\sum_\lambda \varepsilon^\mu_\lambda \varepsilon^\nu_\lambda \leftrightarrow -\left[g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}\right]$$

$$p^2 = m^2$$