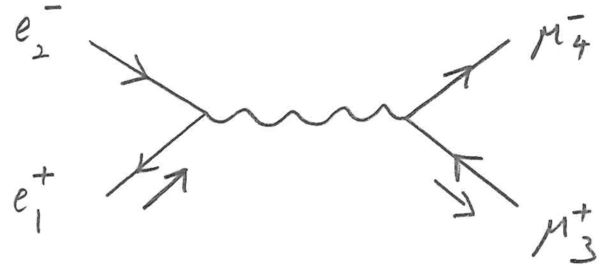
03.2025

Important QED processes

①

$e^+ e^- \rightarrow \mu^+ \mu^-$

$\;\;1\;\;\;2\qquad\;\;\, 3\;\;\;4$



→ time

$$i\mathcal{M} = \left[\bar{v}_1 (-ie\gamma^\mu) u_2\right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{u}_4 (-ie\gamma^\nu) v_3\right]$$

$1 \leftrightarrow p_1\, s_1$ collective index

$$q = p_1 + p_2 = p_3 + p_4$$

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} \left| \bar{v}_1 \gamma^\mu u_2 \; \bar{u}_4 \gamma_\mu v_3 \right|^2$$

$$= \frac{e^4}{q^4} \left(\bar{v}_1 \gamma^\mu u_2 \; \bar{u}_2 \gamma^\nu v_1 \right) \left(\bar{u}_4 \gamma_\mu v_3 \; \bar{v}_3 \gamma_\nu u_4 \right)$$

w. $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$

Casimir's trick:
sum over final states
averaged over initial states

$$\frac{1}{4} \sum_{s_1 s_2} \sum_{s_3 s_4} |\mathcal{M}|^2_{[s_1 s_2 s_3 s_4]}$$

$\updownarrow$ $\langle |\mathcal{M}|^2 \rangle$

$$\sum_{s_1} v_1 \bar{v}_1 = \frac{\not{p}_1 - m_1}{2m_1}$$

$$\sum_{s_2} u_2 \bar{u}_2 = \frac{\not{p}_2 + m_2}{2m_2}$$

$$\langle |M|^2 \rangle \to \frac{1}{4} \frac{e^4}{q^4} t_1 \cdot t_2 \left(\prod_i \frac{1}{2m_i} \right)$$

$$t_1 = tr\left[(\not{p}_1 - m_1) \gamma^\mu (\not{p}_2 + m_2) \gamma^\nu \right]$$

$$t_2 = tr\left[(\not{p}_4 + m_4) \gamma_\mu (\not{p}_3 - m_3) \gamma_\nu \right]$$

$$m_1, m_2 = m_e$$
$$m_3, m_4 = m_\mu$$

$$t_1 \to 4 \left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1 \cdot p_2) g^{\mu\nu} - m_e^2 g^{\mu\nu} \right)$$

$$t_2 \to 4 \left(p_{3\mu} p_{4\nu} + p_{3\nu} p_{4\mu} - (p_3 \cdot p_4) g_{\mu\nu} - m_\mu^2 g_{\mu\nu} \right)$$

$$t_1 \cdot t_2 \to 16 \times$$

$$\{ 2\, p_1 \cdot p_3\ p_2 \cdot p_4 + 2\, p_1 \cdot p_4\ p_2 \cdot p_3 +$$
$$+ 2 m_\mu^2 (p_1 \cdot p_2)^2 + 2 m_e^2 (p_3 \cdot p_4)^2 + 4 m_e^2 m_\mu^2 \}$$

check all the signs !

$$\langle |M|^2 \rangle \longrightarrow 8 \frac{e^4}{\delta^4} \prod_i \frac{1}{2m_i} \times$$

$$\{ P_1 \cdot P_3 \; P_2 \cdot P_4 + P_1 \cdot P_4 \; P_2 \cdot P_3 +$$

$$m_e^2 (P_3 \cdot P_4) + m_\mu^2 (P_1 \cdot P_2) + 2 m_e^2 m_\mu^2 \}$$

CM frame

$$\left. \begin{array}{ll} P_1 : & E, \vec{p}_1 \\ P_2 : & E, -\vec{p}_1 \\ P_3 : & E, \vec{P}_3 \\ P_4 : & E, -\vec{P}_3 \end{array} \right\} \quad \begin{array}{l} P_1 \cdot P_3 = E^2 - \vec{p}_1 \cdot \vec{p}_3 \\ P_2 \cdot P_4 = P_1 \cdot P_3 \\ P_1 \cdot P_4 = E^2 + \vec{p}_1 \cdot \vec{p}_3 \\ P_2 \cdot P_3 = P_1 \cdot P_4 \end{array}$$

$$P_1 \cdot P_2 = E^2 + \vec{p}_1^{\,2} = 2E^2 - m_e^2$$

$$P_3 \cdot P_4 = E^2 + \vec{p}_3^{\,2} = 2E^2 - m_\mu^2$$

$$\sqrt{s} = 2E \qquad |\vec{p}_1| = \sqrt{E^2 - m_e^2}$$

$$|\vec{p}_3| = \sqrt{E^2 - m_\mu^2}$$

we see ...

$\langle |M|^2 \rangle$ depends only on

$$[E, \hat{p}_1 \cdot \hat{p}_3 = \cos\theta]$$

$$\langle |M|^2 \rangle = 8 \frac{e^4}{16 E^4} \times \left(\prod_i \frac{1}{2m_i} \right) \times$$

$$\Big\{ (E^2 - p_1 p_3 \cos\theta)^2 + (E^2 + p_1 p_3 \cos\theta)^2 + (2E^2 - m_\mu^2) M_e^2 + (2E^2 - M_e^2) m_\mu^2 + 2M_e^2 M_\mu^2 \Big\}$$

$$= e^4 \prod_i \frac{1}{2m_i} \times \left\{ \left(1 + \frac{M_e^2 + M_\mu^2}{E^2}\right) + \left(1 - \frac{M_e^2}{E^2}\right)\left(1 - \frac{M_\mu^2}{E^2}\right) \cos^2\theta \right\}$$

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}} \, d\Phi_2 \, \langle |M|^2 \rangle \, f_s$$

see cal. notes.

$$\frac{1}{4} \frac{1}{|\vec{p}_1| \sqrt{s}}$$

see cal. notes.

$$\frac{|\vec{P}_3|}{4\pi\sqrt{s}} \frac{d\Omega}{4\pi}$$

special factor

$$\rightarrow \prod_i 2m_i$$

for this convention.

$$\sqrt{s} = 2E$$

Donoghue et al. Appendix C.

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_3|}{|\vec{p}_1|} \sqrt{s} \langle |M|^2 \rangle$$

$$\rightarrow \frac{\alpha^2}{4s} \frac{\sqrt{1-\frac{4m_\mu^2}{s}}}{\sqrt{1-\frac{4m_e^2}{s}}} \times$$

$$\left\{ \left(1 + \frac{4(m_e^2+m_\mu^2)}{s}\right) + \left(1-\frac{4m_e^2}{s}\right)\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta \right)$$

$$\sigma = \frac{4\pi\alpha^2}{3s} \frac{\sqrt{1-\frac{4m_\mu^2}{s}}}{\sqrt{1-\frac{4m_e^2}{s}}} \left\{ 1 + \frac{1}{2}\frac{4(m_e^2+m_\mu^2)}{s} + \frac{1}{4}\frac{4m_e^2}{s}\frac{4m_\mu^2}{s} \right\}$$

if $m_e \rightarrow 0$

$$\sigma \rightarrow \underline{\frac{4\pi\alpha^2}{3s}} \; \underline{\sqrt{1-\frac{4m_\mu^2}{s}} \left\{ 1 + \frac{1}{2}\frac{4m_\mu^2}{s} \right\}}.$$

$$\simeq \left[1 - \frac{3}{8}\left(\frac{4m_\mu^2}{s}\right)^2 \right]$$

$\frac{4\pi}{3}\frac{\alpha^2}{s}$

is a famous result!

dim. analysis!

for $s \gg 4m_\mu^2$

leading correction!

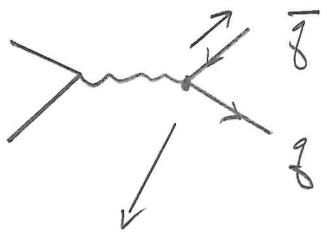
<u>Counting Quarks</u>

$\sqrt{s} \to \infty$

$$\sigma_{e^+e^- \to f\bar{f}} \to \frac{4\pi}{3}\frac{\alpha^2}{s}$$

$$e^+e^- \to u\bar{u},\ d\bar{d}$$
$$\to s\bar{s},\ c\bar{c},\ \text{etc.}\ ?$$

what would change?



$-ie Q_q \gamma^\mu$

$$\sigma_{e^+e^- \to q\bar{q}} \to Q_q^2 N_c \frac{4\pi}{3}\frac{\alpha^2}{s}$$

$$\sigma_{e^+e^- \to \forall q\bar{q}} \to \sum_i \left(Q_i^2 N_c\right) \frac{4\pi}{3}\frac{\alpha^2}{s}$$

$\to$ count only ~~those flavor~~ which satisfies

$$\sqrt{s} > 2m_i$$

$$R := \frac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to \mu^+\mu^-}}$$

$$\to 3 \times \sum_i Q_i^2\ \theta\left(\sqrt{s} - 2m_i\right)$$

for $\sqrt{s} \gg 1$ GeV

u, d, s are all excited

$$R_3 \to 3 \times \left\{ \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \right\}$$

$$\simeq 2$$

if c is excited ($M_c \sim 1.28$ GeV)

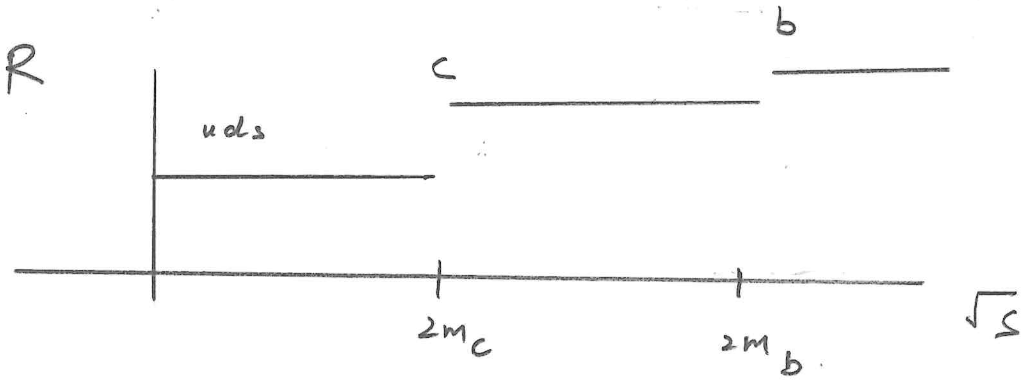
$$R_4 \sim 2 + 3 \times \left(\frac{2}{3}\right)^2$$

$$\simeq 3.33.$$

if even b is around ($\sqrt{s} > 8.4$ GeV)

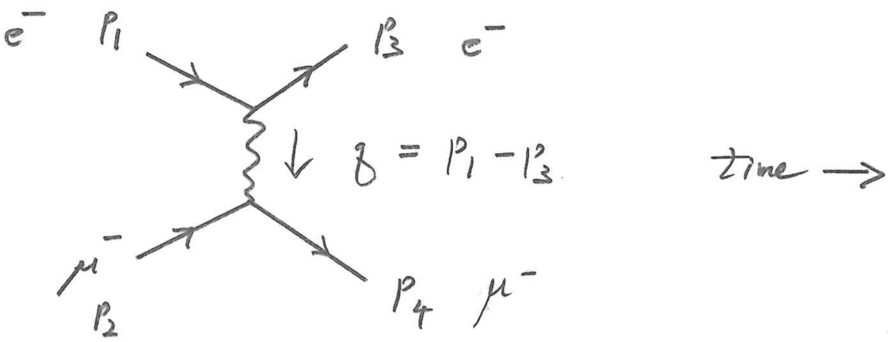
($M_b = 4.2$ GeV)

$$R \simeq 3.66.$$



$\longrightarrow$ Amazing piece of physics ...

$e^- \mu^-$ scattering



$$i\mathcal{M} = \frac{i e^2}{q^2} \bar{u}_3 \gamma^\mu u_1 \quad \bar{u}_4 \gamma_\mu u_2$$

$$\langle |\mathcal{M}|^2 \rangle \rightarrow \frac{e^4}{4q^2} \prod_i \frac{1}{2m_i} \quad t_1 \cdot t_2$$

$\frac{1}{4}\sum_{s s'}$

$$t_1 = tr\left[(\not{p}_1 + m_1)\gamma^\mu (\not{p}_3 + m_3)\gamma^\nu\right]$$

$$t_2 = tr\left[(\not{p}_4 + m_4)\gamma_\mu (\not{p}_2 + m_2)\gamma_\nu\right]$$

$$m_1 = m_3 = m_e$$

$$m_2 = m_4 = m_\mu$$

$$t_1 \cdot t_2 = 16 \times 2 \times$$

$$\{ p_1 \cdot p_4 \; p_3 \cdot p_2 + p_1 \cdot p_2 \; p_3 \cdot p_4$$

$$- p_2 \cdot p_4 \; m_e^2 - p_1 \cdot p_3 \; m_\mu^2$$

$$+ 2 m_e^2 m_\mu^2 \}.$$

$$\langle |M|^2 \rangle \rightarrow 8 \frac{e^4}{q^4} \prod_i \frac{1}{2m_i} \times \{$$

$$p_1 \cdot p_4 \; p_3 \cdot p_2 + p_1 \cdot p_3 \; p_3 \cdot p_4$$

$$- m_e^2 \, p_2 \cdot p_4 - m_\mu^2 \, p_1 \cdot p_3 + 2 m_e^2 m_\mu^2 \}.$$

Again go to the cm frame

$p_1 : (\varepsilon_1 \; \vec{p}_1)$

$p_2 : (\varepsilon_2 \; -\vec{p}_1)$

$p_3 : (\varepsilon_3 , \vec{p}_3)$

$p_4 : (\varepsilon_4 , -\vec{p}_3)$

$\varepsilon_1 = \sqrt{\vec{p}_1^{\,2} + m_e^2}$

$\varepsilon_2 = \sqrt{\vec{p}_1^{\,2} + m_\mu^2}$

etc.

for sure : $|\vec{p}_1| = |\vec{p}_3|$

$\varepsilon_1 = \varepsilon_3 \; ; \; \varepsilon_2 = \varepsilon_4$

$\sqrt{s} = \varepsilon_1 + \varepsilon_2$

$$\Rightarrow |\vec{p}_1| = \frac{1}{2} \sqrt{s} \sqrt{1 - \frac{(m_e + m_\mu)^2}{s}} \sqrt{1 - \frac{(m_e - m_\mu)^2}{s}}$$

is a good starting point

$\longrightarrow \varepsilon_1 = \sqrt{\vec{p}_1^{\,2} + m_e^2}$

$\longrightarrow \varepsilon_2 = \sqrt{\vec{p}_1^{\,2} + m_\mu^2}$

other variable : $\cos\theta \longleftrightarrow \hat{p}_1 \cdot \hat{p}_3$

$$P_1 \cdot P_4 = P_2 \cdot P_3 = \varepsilon_1 \varepsilon_2 + p_1^2 \cos\theta$$

$$P_1 \cdot P_3 \neq P_2 \cdot P_4 :$$

$$P_1 \cdot P_3 = \varepsilon_1^2 - p_1^2 \cos\theta$$
$$P_2 \cdot P_4 = \varepsilon_2^2 - p_1^2 \cos\theta$$

$$P_1 \cdot P_2 = P_3 \cdot P_4 = \varepsilon_1 \varepsilon_2 + p_1^2$$

$$q^2 = (P_1 - P_3)^2 = -(\vec{p}_1 - \vec{p}_3)^2 = -2p_1^2(1-\cos\theta)$$

$$\langle |M|^2 \rangle \longrightarrow 8 \frac{e^4}{4 p_1^4 (1-\cos\theta)^2} \prod_i \frac{1}{2m_i} \times$$

$$\left\{ (\varepsilon_1 \varepsilon_2 + p_1^2 \cos\theta)^2 + (\varepsilon_1 \varepsilon_2 + p_1^2)^2 - (m_e^2 + m_\mu^2)\, p_1^2 (1-\cos\theta) \right\}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{s} \frac{p_3}{p_1} \langle |M|^2 \rangle \; f_s$$

$$f_s \rightarrow \prod_i 2m_i$$

$$\longrightarrow \frac{\alpha^2}{2} \frac{1}{s} \frac{1}{(1-\cos\theta)^2} \times$$

$$\left\{ \left(\frac{\varepsilon_1 \varepsilon_2}{p_1^2} + \cos\theta\right)^2 + \left(\frac{\varepsilon_1 \varepsilon_2}{p_1^2} + 1\right)^2 - (m_e^2 + m_\mu^2) \frac{1}{p_1^2} (1-\cos\theta) \right\}.$$

if we take $m_e \to 0$

$$p_1 \to \frac{1}{2}\sqrt{s}\left(1 - \frac{M_\mu^2}{s}\right)$$

$$E_1 \to p_1$$

$$\frac{d\sigma}{d\Omega} \to \frac{\alpha^2}{2}\frac{1}{s}\frac{1}{(1-\cos\theta)^2} \times$$

$$\left\{\left(\frac{E_2}{p_1} + \cos\theta\right)^2 + \left(\frac{E_2}{p_1}+1\right)^2 - M_\mu^2\frac{1}{p_1^2}(1-\cos\theta)\right\}$$

if even $M_\mu \to 0$

$$\frac{d\sigma}{d\Omega} \to \frac{\alpha^2}{2}\frac{1}{s}\frac{1}{(1-\cos\theta)^2} \times$$

$$\left\{4+(1+\cos\theta)^2\right\}$$

$$\theta \to 0 \qquad \frac{d\sigma}{d\Omega} \to \infty$$

the famous IR div problem!

extra note: spinor convention

$$\psi_x = \sum_s \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{m}{E_p}} \left[u_{ps}\, e^{-ip\cdot x}\, b_{\vec{p}s} + v_{ps}\, e^{ip\cdot x}\, d^\dagger_{\vec{p}s} \right]$$

$$\{ b_{ps}, b^\dagger_{p's'} \} = (2\pi)^3 \delta^3_{\vec{p}-\vec{p}'}\, \delta_{ss'}$$

similarly for d

$$\sum_s u_{ps}\bar{u}_{ps} = \frac{\not{p}+m}{2m}$$

$\bar{u}u = \delta_{ss'}$ $\qquad$ $\bar{v}v = -\delta_{ss'}$

$u^\dagger u = \frac{E}{m}\delta_{ss'}$ $\qquad$ $v^\dagger v = \frac{E}{m}\delta_{ss'}$

$\{\psi, \psi^\dagger\} = \delta^3_{xy}$ is properly normalized

✓ good.

<u>but</u> $f_s \longrightarrow \prod_i 2m_i$

vs the other convention:

$$\psi_x = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(\tilde{u}\, e^{-ip\cdot x}\, \tilde{b} + \tilde{v}\, e^{ip\cdot x}\, \tilde{d}^\dagger \right)$$

$$\sum u\tilde{\bar{u}} = (\not{p}+m).$$

$$\{\tilde{b}_{\vec{p}s}, \tilde{b}^{\dagger}_{\vec{p}'s'}\} \longrightarrow (2\pi)^3 \delta^3_{\vec{p}-\vec{p}'} \delta_{ss'} \, 2E_p$$

$$\tilde{u} \longleftrightarrow \sqrt{2m}\, u$$

$$\tilde{b} \longleftrightarrow \sqrt{2E_p}\, b$$

$\Longrightarrow$ $\sqrt{\dfrac{m}{E_p}}$ is needed for ψ

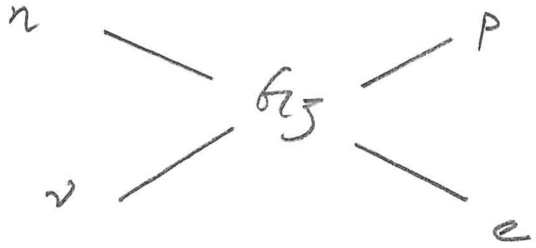
& No need $\prod_i 2m_i$ for S-matrix element

2025

Lessons from Fermi's theory ①

$$\mathcal{L} = -\frac{1}{\sqrt{2}} G_F \; \bar{p}\, \Gamma^{\mu} n \; \bar{e}\, \Gamma_{\mu} \nu$$

4-fermion theory

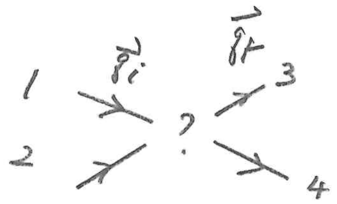


$n \to p + e^- + \bar{\nu}$

G_F : 1.166×10^{-5} GeV^{-2}.

① Non-relativistic formulation

Fermi Golden Rule



$$\frac{d\sigma}{d\Omega} = 2\pi \, |M|^2 \, \frac{1}{V_i} \, \underline{\frac{1}{(2\pi)^3} \, q_f^2 \, \frac{dq_f}{d\sqrt{s}}}$$

at a given E or $\sqrt{s}$

density of final states

$$N_f = V \int \frac{d^3 q_f}{(2\pi)^3}$$

$$\frac{d^2 N_f}{d\sqrt{s}\, d\Omega}$$

CM frame

$$\sqrt{s} = \sqrt{q_i^2 + m_1^2} + \sqrt{q_i^2 + m_2^2}$$

$$= \sqrt{q_f^2 + m_3^2} + \sqrt{q_f^2 + m_4^2}$$

$$v_i = \frac{d\sqrt{s}}{d q_i} = \frac{q_i}{\varepsilon_1} + \frac{q_i}{\varepsilon_2}$$

$$= \frac{q\sqrt{s}}{\varepsilon_1 \varepsilon_2}$$

$$v_f = \frac{q_f \sqrt{s}}{\varepsilon_3 \varepsilon_4}$$

if $m_3, m_4 \to 0$

$\sqrt{s} \to 2 q_f \Rightarrow v_f \to 2$ / non-sense! need Relativistic formulation

$\overleftarrow{c} \; \overrightarrow{c}$

Decay formula

$$\frac{d\Gamma}{d\Omega} = 2\pi \, |M|^2 \frac{1}{(2\pi)^3} q_f^2 \frac{d q_f}{d\sqrt{s}}$$

$\nu_e \; e \longrightarrow \nu_e \; e$

Perkins
p. 152

$$\frac{d\sigma}{d\Omega} = (2\pi)\,|M|^2\,\frac{1}{v_i}\,\frac{1}{(2\pi)^3}\,q_f^2\,\frac{dq_f}{d\sqrt{s}}\quad 2$$

sum over final states
av. over initial
(for e^-)

ν : LH
by V−A

$m_\nu \to 0$

$m_e \approx 0$

$v_i \approx 2$

$$M \to \frac{1}{2}\frac{g_W^2}{M_W^2}$$

$$\parallel$$

$$\frac{4G_F}{\sqrt{2}}$$

$\sqrt{s} \approx 2q_f$

$$\frac{dq_f}{d\sqrt{s}} = \frac{1}{2}$$

$$\frac{d\sigma}{d\Omega} \to 2\pi\,\frac{g_W^4}{4M_W^4}\,\frac{1}{2}\,\frac{1}{8\pi^3}\,q_f^2\,\frac{1}{2}\,(2)$$

$$= \frac{g_W^4}{32M_W^4}\,\frac{1}{\pi^2}\,q_f^2$$

$$\frac{1}{2}G_F^2 \leftrightarrow \left(\frac{1}{8}\frac{g_W^2}{M_W^2}\right)^2$$

$$\frac{d\sigma}{d\Omega} = G_F^2 \frac{q_f^2}{\pi^2}$$

$$\sigma \to G_F^2 \frac{1}{\pi} \left(4q_f^2\right)$$

$$\sim$$

$$\sigma \quad G_F^2 \frac{1}{\pi} s. \quad //$$

Energetic ν is easier to detect ..

Nice result.

Perkins 5.24.

see the QFT derivation !

② UV catastrophe.

$$\sigma \sim G_F^2 \frac{s}{\pi} \to \infty \quad \text{as} \quad s \to \infty.$$

$\to$ signal for new physics : W-bosons.

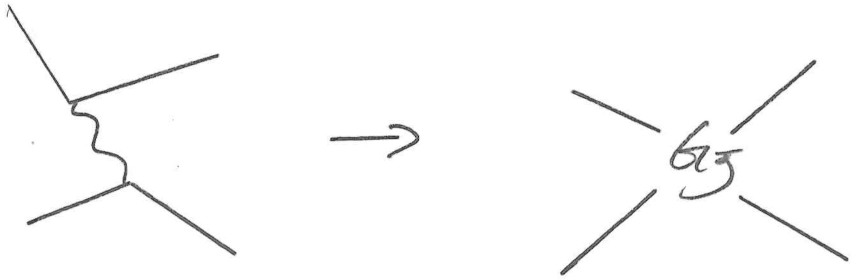
indeed

$$\sigma \to G_F^2 \frac{M_W^2}{\pi} \quad \text{as} \quad s \to \infty$$

after a correct calculation.

see notes : weak process.

3 effective field theory



coupling $\longleftrightarrow$ integrate out heavy fields

4-fermion int is not renormalizable

$+ \cdots$

perturbative series of G_F.

$$\tilde{G}_F = G_F \left(1 + G_F \, l + G_F^2 \, l^2 + \cdots \right)$$

l = loops.

$l \sim \Lambda^2$ by dim. analysis.

more & more UV if we go further

DIV is not solved!

if we don't know the full theory

integrate out many scales

low E theorem
couplings etc. $\longrightarrow$ χPT BSM.
NRL ...

what kind of terms
are allowed in the
Lagrangian? $\longrightarrow$ Symmetry
dictates
a lot ...

for weak physics.
full QFT is known.

$\bar{\psi}\gamma^{\mu}(1-\gamma_5)\psi$ Dirac structure.

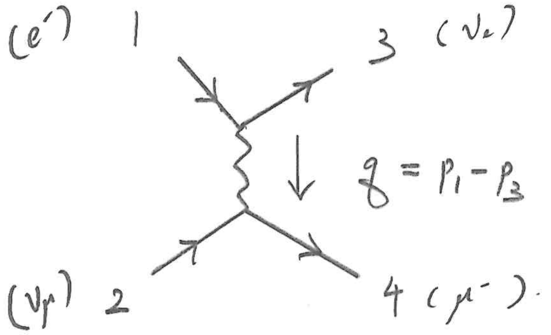
V$-$A $\psi \rightarrow$ spinors.

Gauge symmetries : local
global $\rightarrow$ SB.

Weak Processes

ref. Griffiths

$$\overset{1}{e^-} + \overset{2}{\nu_\mu} \longrightarrow \overset{3}{\nu_e} + \overset{4}{\mu^-}$$



$$iM = \frac{-i}{q^2 - M_W^2}\left(\frac{-i g_W}{2\sqrt{2}}\right)^2 \bar{u}_3 \gamma^\mu (1-\gamma_5) u_1 \; \bar{u}_4 \gamma_\mu (1-\gamma_5) u_2$$

$$M \approx -\frac{g_W^2}{8 M_W^2} \bar{u}_3 \gamma^\mu (1-\gamma_5) u_1 \; \bar{u}_4 \gamma_\mu (1-\gamma_5) u_2$$

$$\langle |M|^2 \rangle \longrightarrow \frac{1}{2} \sum_{s_1 s_3} |M|^2$$

sum over finals
average over initial
→ fac is for e^-
$\nu \to \nu_L$ always left-handed

$$\langle |M|^2 \rangle = \frac{1}{2}\left(\frac{g_W^2}{8 M_W^2}\right)^2 t_1 \cdot t_2 \quad \textstyle\prod (2m_i)^{-1}$$

↳ not for ν's

$$t_1 \rightarrow tr\left[\gamma^\mu(1-\gamma_5)(\not{p}_1+m_e)\gamma^\nu(1-\gamma_5)\not{p}_3\right]$$

$$t_2 \rightarrow tr\left[\gamma_\mu(1-\gamma_5)\not{p}_2\gamma_\nu(1-\gamma_5)(\not{p}_4+m_\mu)\right]$$

$$t_1 = tr\left[\gamma^\mu(1-\gamma_5)(\not{p}_1+m_e)\gamma^\nu(1-\gamma_5)\not{p}_3\right]$$

① m_e term $\rightarrow 0$

$$\sim (1-\gamma_5)(1+\gamma_5) \rightarrow 0$$

$$\rightarrow tr\left[\gamma^\mu \underbrace{(1-\gamma_5)^2}_{2(1-\gamma_5)} \not{p}_1 \gamma^\nu \not{p}_3\right]$$

$$= 2\,tr\left[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3\right] + 2\,tr\left[\gamma_5 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3\right]$$

$$= 8\left\{ p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - (p_1\cdot p_3)g^{\mu\nu} + i\varepsilon^{\mu\nu\alpha\beta} p_{1\alpha} p_{3\beta} \right\}$$

similarly

$$t_2 \rightarrow 8\left\{ p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - (p_2\cdot p_4)g_{\mu\nu} + i\varepsilon_{\mu\nu\alpha\beta} p_2^\alpha p_4^\beta \right\}.$$

to get these results, we use

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Peskin & Schröder convention p. 134

$$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\,\varepsilon^{\mu\nu\rho\sigma}$$

to show that, argue LHS $\sim \varepsilon^{\mu\nu\rho\sigma}$, figure out the constant by computing

$$\mathrm{tr}(i\gamma^0\gamma^1\gamma^2\gamma^3\;\gamma^0\gamma^1\gamma^2\gamma^3)$$

$$= -i\,\mathrm{tr}\,\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3$$

$$= -4i$$

$\varepsilon^{\mu\nu\rho\sigma} = 1$ if $\mu\nu\rho\sigma \sim 0123$

$\longrightarrow$ permutations induce -1

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\rho\sigma} = -\underline{2}\,(\delta^\alpha{}_\rho\,\delta^\beta{}_\sigma - \delta^\alpha{}_\sigma\,\delta^\beta{}_\rho)$$

relative − sign

2 way to sum $\mu\nu$

Also $\left|\bar{u}_3\gamma^\mu(1-\gamma_5)u_1\right|^2$

$$= \bar{u}_3\gamma^\mu(1-\gamma_5)u_1\;\; u_1^\dagger(1-\gamma_5)\gamma^0\gamma^\nu\gamma^0\gamma^0 u_3$$

$$\rightarrow \mathrm{tr}\left[\gamma^\mu(1-\gamma_5)\;u\bar{u}_1\;\gamma^\nu(1-\gamma_5)\,u_3\bar{u}_3\right]$$

$$t_1 \cdot t_2 = 64 \times \{$$

$$2\,(p_1 \cdot p_2 \;\; p_3 \cdot p_4 + p_1 \cdot p_4 \;\; p_3 \cdot p_2)$$

$\varepsilon\,\varepsilon$ term.

$$+ \; 2\,(p_1 \cdot p_2 \;\; p_3 \cdot p_4 - p_1 \cdot p_4 \;\; p_2 \cdot p_3) \}.$$

w huge cancellation of

$p_1 \cdot p_3 \;\; p_2 \cdot p_4$

$$\Rightarrow \; t_1 \cdot t_2 = 64 \times 4 \times p_1 \cdot p_2 \;\; p_3 \cdot p_4$$

$$\langle |M|^2 \rangle = \frac{1}{2} \, \frac{g_W^4}{64\, M_W^4} \, t_1 \cdot t_2 \prod 2m_i^{-1}$$

$$\longrightarrow \frac{1}{2} \, \frac{g_W^4}{M_W^4} \, \underline{4\,(p_1 \cdot p_2)(p_3 \cdot p_4)} \, \prod_i 2m_i^{-1}$$

beautifully simple!

$\sqrt{s}^{-1}$

to be cancelled later

$p_1 : (\varepsilon_1, \vec{p}_1)$

$p_2 : (|\vec{p}_1|, -\vec{p}_1)$

$p_3 : (|\vec{p}_2|, \vec{p}_3)$

$p_4 : (\varepsilon_4, -\vec{p}_3)$

$$\varepsilon_1 = \sqrt{\vec{p}_1^{\,2} + m_e^2}$$

$$\varepsilon_4 = \sqrt{\vec{p}_3^{\,2} + m_\mu^2}$$

$$s = (\varepsilon_1 + \underbrace{|\vec{p}_1|}_{q_i})^2 = (\varepsilon_2 + \underbrace{|\vec{p}_2|}_{q_f})^2$$

$$\Downarrow$$

$$q_i = \frac{1}{2}\sqrt{s}\left(1 - \frac{m_e^2}{s}\right)$$

$$q_f = \frac{1}{2}\sqrt{s}\left(1 - \frac{m_\mu^2}{s}\right)$$

$$p_1 \cdot p_2 = \frac{1}{2}\left[(p_1+p_2)^2 - p_1^2 - p_2^2\right]$$

$$= \frac{1}{2}(s - m_e^2)$$

$$p_3 \cdot p_4 = \frac{1}{2}(s - m_\mu^2)$$

Recall :

$$d\sigma = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - p_1^2 p_2^2}} \langle |M|^2 \rangle \; d\Phi_2 \; f_S$$

$$= \frac{1}{4\sqrt{q_i^2 s}} \langle |M|^2 \rangle \frac{q_f}{4\pi\sqrt{s}} \frac{1}{4\pi} d\Omega \; f_S$$

$$= \frac{1}{64\pi^2} \frac{q_f}{q_i} \frac{1}{s} \langle |M|^2 \rangle f_S \; d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{g_W^4}{M_W^4} \, 2 \, q_f^2$$

W $\quad q_f = \frac{1}{2}\sqrt{s}\left(1 - \frac{m_\mu^2}{s}\right).$

identify

$$\frac{1}{2} G_F^2 = \left(\frac{g_W^2}{8M_W^2}\right)^2$$

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{\pi^2} q_f^2$$

//

Story Time :

the same formula describes

$e\,\nu_e \to e\,\nu_e$. if $m_\mu \to m_e$.

forget m_e

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{\pi^2} \frac{1}{4} s.$$

$$\sigma \to G_F^2 \frac{1}{\pi} s$$

Perkins 5.24.

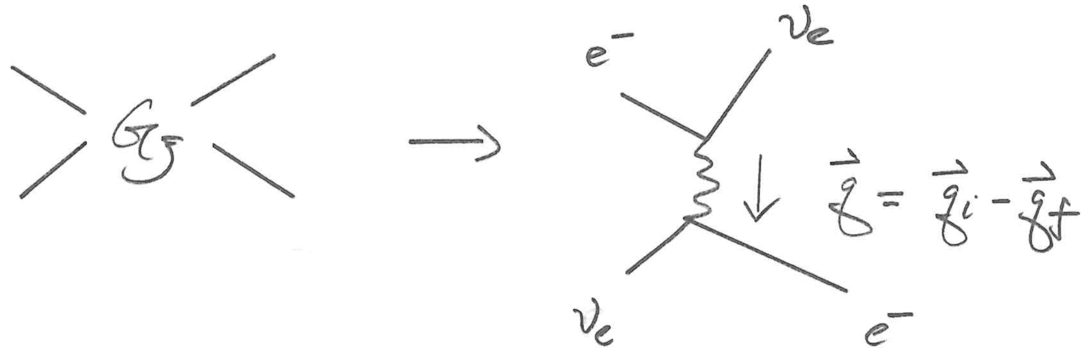
Nice result $\to$ dim. analysis.

$\hookrightarrow$ high E ν's are easier to detect e.g. Homestake. exp.

but if $s \to \infty$ $\sigma \to \infty$

$\Rightarrow$ UV catastrophe

what's wrong!



W-boson!

Discovery has to happen.

$$q^2 = (E_e - E_{\nu_e})^2 - (\vec{q}_i - \vec{q}_f)^2$$

$E_e - E_{\nu_e} \to 0$ for massless m_e.

$|\vec{q}_i| = |\vec{q}_f|$.

$$\to -2 q_f^2 (1 - \cos\theta).$$

instead of

$$\frac{g_W^4}{M_W^4} \to \frac{g_W^4}{(q^2 - M_W^2)^2}$$

$$= \frac{g_W^4}{M_W^4} \left(\frac{-M_W^2}{q^2 - M_W^2} \right)^2$$

$$\to \frac{g_W^4}{M_W^4} \left\{ \frac{1}{1 + \frac{2 g_f^2}{M_W^2}(1 - \cos\theta)} \right\}^2$$

the ratio of σ :

$$r = \frac{\int_{-1}^{1} dz \left\{ \frac{1}{1 + \frac{2 g_f^2}{M_W^2}(1-z)} \right\}^2}{\int_{-1}^{1} dz \, (1)}$$

$$= \frac{1}{2} \int_0^2 dz' \left(\frac{1}{1 + \frac{2 g_f^2}{M_W^2} z'} \right)^2$$

$$\to \frac{1}{1 + \frac{4 g_f^2}{M_W^2}} \quad //$$

$g_f \to 0 \qquad r \to 1 \qquad$ old result.

$$g_f \to \infty \qquad r \to \frac{M_W^2}{4 g_f^2} = \frac{M_W^2}{s}$$

$$s \to \infty$$

$$\sigma : \frac{G_F^2}{\pi} s \, r \to \frac{G_F^2}{\pi} M_W^2$$

just replace.

$s \to M_W^2$.

No div. at all

the world is saved

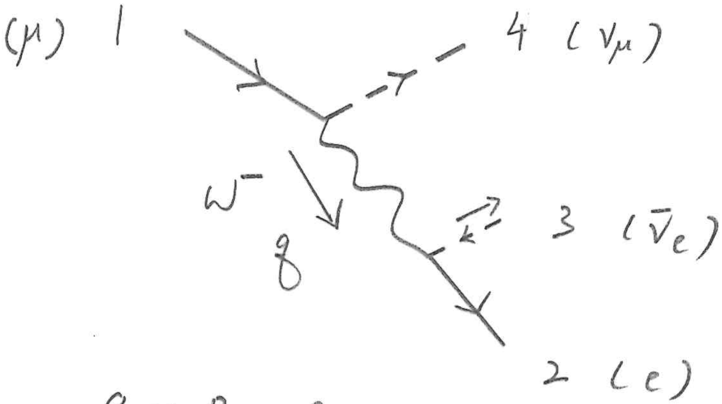
Muon decay

$\mu \rightarrow e \quad \bar{\nu}_e \quad \nu_\mu$

$1 \qquad 2 \quad 3 \quad 4$

CM frame =
rest frame of μ

$P_1 = P_2 + P_3 + P_4$



$q = P_1 - P_4$

$$\mathcal{M} = -\frac{g_W^2}{8 M_W^2} \, \bar{u}_4 \gamma^\mu (1-\gamma_5) u_1 \;\; \bar{u}_2 \gamma_\mu (1-\gamma_5) v_3$$

$$\langle |\mathcal{M}|^2 \rangle \longrightarrow \frac{1}{2} \frac{g_W^4}{64 M_W^4} \, t_1 \cdot t_2 \qquad f_c^{-1}$$

$$t_1 = tr\left[\gamma^\mu (1-\gamma_5)(\not{p}_1 + m_\mu)\gamma^\nu (1-\gamma_5) \not{p}_4 \right]$$

$$t_2 = tr\left[\gamma_\mu (1-\gamma_5) \not{p}_3 \, \gamma_\nu (1-\gamma_5)(\not{p}_2 + m_e)\right]$$

m_e, m_μ terms X contribute :

$$t_1 = 8 \{ p_1^\mu p_4^\nu + p_1^\nu p_4^\mu - g^{\mu\nu} (p_1 \cdot p_4)$$
$$+ i \varepsilon^{\mu\nu\alpha\beta} p_{1\alpha} p_{4\beta} \}.$$

$$t_2 = 8 \{ p_{3\mu} p_{2\nu} + p_{3\nu} p_{2\mu} - g_{\mu\nu} p_3 \cdot p_2$$
$$+ i \varepsilon_{\mu\nu\alpha\beta} p_3^\alpha p_2^\beta \}.$$

$$t_1 \cdot t_2 = 64 \; (4) \; p_1 \cdot p_3 \; p_2 \cdot p_4$$

$$p_1 = (m_\mu, \vec{0}) \qquad p_1 = p_2 + p_3 + p_4$$

$$p_1 \cdot p_3 = m_\mu \varepsilon_3 .$$

$$p_2 \cdot p_4 = \frac{1}{2} \left[(p_2 + p_4)^2 - p_2^2 - p_4^2 \right]$$
$$= \frac{1}{2} \left((p_1 - p_3)^2 - m_e^2 \right).$$
$$= \frac{1}{2} \left[m_\mu^2 - 2 m_\mu \varepsilon_3 - m_e^2 \right]$$

$$\langle |M|^2 \rangle = \frac{g_W^4}{M_W^4} m_\mu E_{\bar{\nu}_e} \left[m_\mu^2 - 2 m_\mu E_{\bar{\nu}_e} - m_e^2 \right] \quad f_s^{-1}$$

$$d\Gamma_{\mu \to e \bar{\nu}_e \nu_\mu} = \frac{1}{2M_\mu} \langle |M|^2 \rangle \, d\Phi_3$$

$$\downarrow$$

$$\frac{d^3p_2 \, d^3p_3 \, d^3p_4}{[(2\pi)^3]^3} \frac{1}{2E_2 \, 2E_3 \, 2E_4} (2\pi)^4 \delta^4 \begin{matrix} m_\mu - E_2 - E_3 - E_4 \\ \vec{0} - \vec{p}_2 - \vec{p}_3 - \vec{p}_4 \end{matrix}$$

$$\downarrow$$

$$\frac{1}{2m_\mu} \frac{g_W^4}{M_W^4} \times$$

$$\int d\Phi_3 \; m_\mu \overset{E_3}{\overset{\updownarrow}{E_{\bar{\nu}_e}}} \left\{ m_\mu^2 - m_e^2 - 2m_\mu \overset{E_3}{\overset{\updownarrow}{E_{\bar{\nu}_e}}} \right\}$$

(tedious calculation ...)

$$\downarrow$$

$$\Gamma = \frac{g_W^4}{M_W^4} \frac{m_\mu^5}{24(256)\pi^3} = \frac{G_F^2 m_\mu^5}{192 \pi^3} \quad //$$

note $\frac{1}{2} G_F^2 \leftrightarrow \frac{g_W^4}{64 M_W^4}$

some details (see cal. notes)

$$\Gamma = \frac{1}{2m_\mu} \frac{g_W^4}{M_W^4} \times \frac{1}{32\pi^3}$$

$$\int d\varepsilon_2 \, d\varepsilon_3 \quad m_\mu \varepsilon_3 \left\{ m_\mu^2 - m_e^2 - 2m_\mu \varepsilon_3 \right\}$$

ε_2 : m_e to $\dfrac{s + m_e^2}{2\sqrt{s}}$

note: $s = m_\mu^2$

ε_3 : $\dfrac{s + m_e^2 - 2\sqrt{s}\,\varepsilon_2}{2\left[\sqrt{s} - \varepsilon_2 \pm \sqrt{\varepsilon_2^2 - m_e^2}\right]}$

if $m_e \to 0$ we can get analytic form

ε_2 : 0 to $\sqrt{s}/2$

ε_3 : $\frac{1}{2}\sqrt{s} - \varepsilon_2$ to $\sqrt{s}/2$

$$\Gamma = \frac{1}{64\pi^3} \frac{g_W^4}{M_W^4} m_\mu^5$$

$$\int_0^{1/2} dx_2 \int_{1/2 - x_2}^{1/2} dx_3 \quad x_3 (1 - 2x_3)$$

$x_2 = \varepsilon_2 / m_\mu$

$x_3 = \varepsilon_3 / m_\mu$

$$\Gamma \to \frac{1}{64\pi^3} \frac{g_W^4}{M_W^4} m_\mu^5 \underbrace{\int_0^{1/2} dx_2 \; \frac{1}{2} x_2^2 \left(1 - \frac{4}{3} x_2\right)}_{1/96}$$

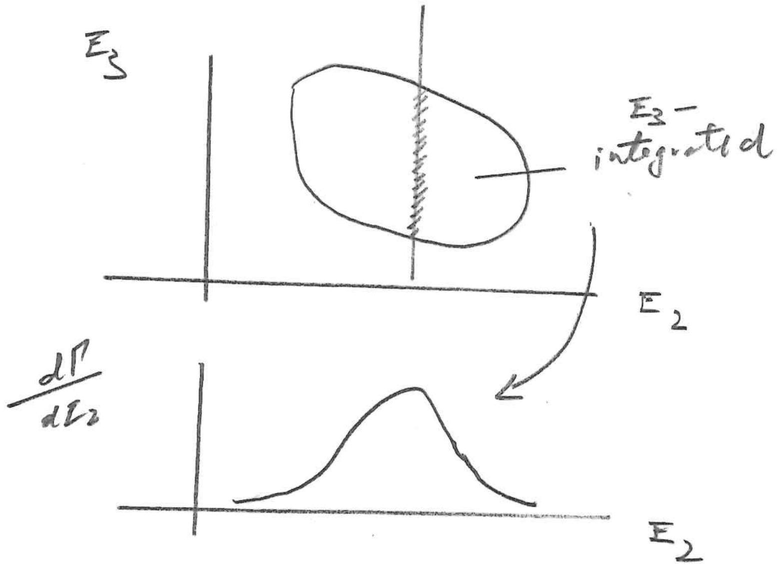
$$= \frac{m_\mu^5}{6144\pi^3} \frac{g_W^4}{M_W^4}$$

$$\updownarrow$$

$$\frac{G_F^2 \, m_\mu^5}{192\pi^3}$$

all but last integral $\to$

$$\frac{d\Gamma}{dE_2} = \frac{1}{128\pi^3} \frac{g_W^4}{M_W^4} m_1^4 \left(\frac{E_2}{m_1}\right)^2 \times \left(1 - \frac{4}{3}\frac{E_2}{m_1}\right).$$



Further Dalitz decay

$\mu \to e \; \bar{\nu}_e \; \nu_\mu$

$n \to p \; \bar{\nu}_e \; e$

essential to take m_e into account

$\swarrow$

$m_n - m_p$; m_e are relevant scales

μ : 2×10^{-6} s

n : 12 min

$\swarrow$ difference is due to <u>kinematics</u>

$\swarrow$

you get 1340 s in this calculation.
(overestimate)

$\swarrow$

n, p's are NOT pointlike ...

$$1 - \gamma_5 \to \underbrace{c_V}_{\simeq 1} - \underbrace{c_A}_{\simeq 1.26} \gamma_5$$

↑ width.

↓ lifetime.

see Num. Demo.

Calculation Notes for UEFT amplitudes

03. 2025

①

$1+2 \to 3+4$

$$d\sigma = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - p_1^2 p_2^2}} \langle |M|^2\rangle \, d\varphi_2^f \, f_s$$

sym. fac. for convention.

$$\frac{1}{4}\frac{1}{\sqrt{q_i^2 s}} \qquad\qquad \frac{q_f}{4\pi\sqrt{s}}\frac{1}{4\pi}d\Omega$$

$$= \frac{1}{64\pi^2}\frac{q_f}{q_i}\frac{1}{s}\langle |M|^2\rangle f_s \, d\Omega$$

a) $(p_1\cdot p_2)^2 - p_1^2 p_2^2 = q_i^2 s$

M1

$$LHS = \left\{\frac{1}{2}\left[(p_1+p_2)^2 - m_1^2 - m_2^2\right]\right\}^2 - m_1^2 m_2^2$$

$$= \frac{1}{4}(s-m_1^2-m_2^2)^2 - m_1^2 m_2^2$$

$$= \frac{1}{4}(s-m_1^2-m_2^2+2m_1m_2)(s-m_1^2-m_2^2-2m_1m_2)$$

$$= s\, q_i^2$$

Where $q_i = \frac{1}{2}\sqrt{s}\sqrt{1 - \frac{(m_1+m_2)^2}{s}}\sqrt{1 - \frac{(m_1-m_2)^2}{s}}$

obtained from inverting:

$$\sqrt{s} = \sqrt{q_i^2 + m_1^2} + \sqrt{q_i^2 + m_2^2}.$$

<u>M2</u>

$$(p_1 \cdot p_2)^2 - p_1^2 p_2^2$$

$$= (\varepsilon_1 \varepsilon_2 + q_i^2)^2 - m_1^2 m_2^2$$

$$= \varepsilon_1^2 \varepsilon_2^2 + 2\varepsilon_1 \varepsilon_2 q_i^2 + q_i^4 - m_1^2 m_2^2$$

$$= 2q_i^4 + q_i^2 \{ m_1^2 + m_2^2 + 2\varepsilon_1 \varepsilon_2 \}$$

$$= q_i^2 \{ \varepsilon_1^2 + \varepsilon_2^2 + 2\varepsilon_1 \varepsilon_2 \}$$

$$= q_i^2 (\varepsilon_1 + \varepsilon_2)^2 = q_i^2 s. \quad //$$

2-body phase space

$$\int d\Phi_2 f = \int \frac{d^3p_3}{(2\pi)^3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{4E_3E_4}(2\pi)^4\delta^4\begin{pmatrix}\sqrt{s}-E_3-E_4\\ \vec{0}-\vec{p}_3-\vec{p}_4\end{pmatrix}$$

$$\longrightarrow \int \frac{d^3p_3}{(2\pi)^3}\frac{1}{4E_3E_4}(2\pi)\,\delta\left(\sqrt{s}-E_3-E_4\right)$$

$$E_4 \longrightarrow \sqrt{\vec{p}_3^{\,2}+m_4^2}$$

$$= \frac{4\pi}{8\pi^3}\int_0^\infty dp_3\,\frac{p_3^2}{4E_3E_4}\,2\pi\,\frac{\delta\left(p_3-q_f(s)\right)}{\frac{\partial}{\partial p_3}(E_3+E_4)}$$

$$= \frac{q_f(s)}{4\pi\sqrt{s}}$$

//

$$q_f(s) = \frac{1}{2}\sqrt{s}\sqrt{1-\frac{(m_3+m_4)^2}{s}}\sqrt{1-\frac{(m_3-m_4)^2}{s}}$$

3-body phase space

$1 \to 2+3+4$

$$\int d\Phi_3 = \int \frac{1}{32\pi^3}\, d\varepsilon_2\, d\varepsilon_3 \qquad \text{Energy form.}$$

$$= \int \frac{1}{32\pi^3} \frac{1}{4s}\, ds_{34}\, ds_{24} \qquad \text{Invar. Mass form.}$$

why ?

$$s_{34} = (p_3+p_4)^2 = (p_1-p_2)^2$$

$$= s + m_2^2 - 2\sqrt{s}\,\varepsilon_2.$$

using $p_1 = (\sqrt{s}, \vec{0})$.

$$ds_{34} = -2\sqrt{s}\, d\varepsilon_2$$

Similarly

$$ds_{24} = -2\sqrt{s}\, d\varepsilon_3.$$

the challenging task is to determine the range of $\{ s_{34}, s_{24} \}$

or $\{ \varepsilon_2, \varepsilon_3 \}$.

$\{ s_{34} , \; s_{24} \}$ basis

1. take s_{34} to be free (integrate last)

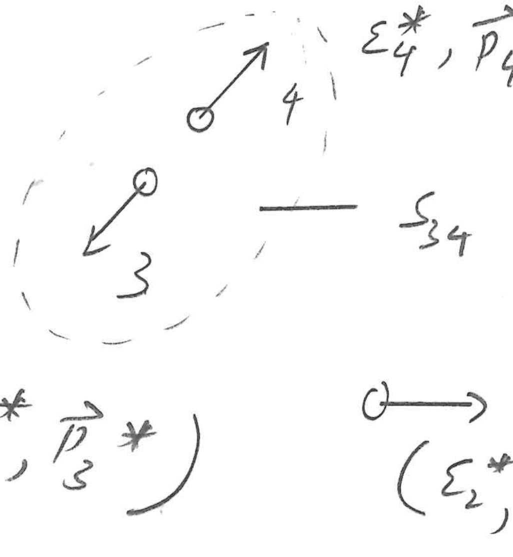
$$s_{34} : \quad (M_3 + M_4)^2 \longrightarrow (\sqrt{s} - M_2)^2$$

massless : $0 \longrightarrow s$.

2. for a given $s_{34} \longrightarrow$ determine the range of s_{24}

the easiest way is to go to the rest frame of $(3,4)$

— the $*$ frame.



$E_4^*, \vec{p}_4^{\,*} = -\vec{p}_3^{\,*}$

s_{34}

$(E_3^*, \vec{p}_3^{\,*})$

$(E_2^*, \vec{p}_2^{\,*})$.

$$s_{24} = (p_2 + p_4)^2 = m_2^2 + m_4^2 + 2\left(E_2^* E_4^* - p_2^* p_4^* z_{24}^*\right)$$

$$s_{24} \underset{\substack{max \\ min}}{\longrightarrow} m_2^2 + m_4^2 + 2\left(E_2^* E_4^* \pm p_2^* p_4^*\right)$$

if we can express

E_2^*, E_4^* as function of s_{34}

$\longrightarrow$ done!

$$p_2^* = \sqrt{E_2^{*2} - m_2^2}$$

$$p_4^* = \sqrt{E_4^{*2} - m_4^2}$$

how ?

$$\begin{aligned} P_{34} \cdot p_4 = \sqrt{s_{34}}\, E_4^* &= (p_3 + p_4) \cdot p_4 \\ &= -\frac{1}{2}\left[(p_3 + p_4 - p_4)^2 - s_{34} - m_4^2\right] \\ &= \frac{1}{2}\left(s_{34} + m_4^2 - m_3^2\right) \end{aligned}$$

$$E_4^* = \frac{1}{2\sqrt{s_{34}}}\left(s_{34} + m_4^2 - m_3^2\right)$$

for $\mathcal{E}_2^*$

$$P_{34} \cdot P_2 = \sqrt{S_{34}}\ \mathcal{E}_2^* = \frac{1}{2}\left[(P_{34}+P_2)^2 - P_{34}^2 - P_2^2 \right]$$

$$\mathcal{E}_2^* = \frac{1}{2\sqrt{S_{34}}} \left(S - S_{34} - m_2^2 \right)$$

$$\int d\Phi_3 \cdots \longrightarrow \int dS_{34} \int dS_{24}\ \frac{1}{32\pi^3} \frac{1}{4S} \cdots$$

$$S_{34} : \ (m_3+m_4)^2 \longrightarrow (\sqrt{S} - m_2)^2$$

$$S_{24} : \ m_2^2 + m_4^2 + 2\left(\mathcal{E}_2^* \mathcal{E}_4^* - \sqrt{\mathcal{E}_2^{*2} - m_2^2}\ \sqrt{\mathcal{E}_4^{*2} - m_4^2} \right)$$

$$\longrightarrow$$

$$m_2^2 + m_4^2 + 2\left(\mathcal{E}_2^* \mathcal{E}_4^* + \sqrt{\mathcal{E}_2^{*2} - m_2^2}\ \sqrt{\mathcal{E}_4^{*2} - m_4^2} \right)$$

$$\mathcal{E}_2^* = \frac{1}{2\sqrt{S_{34}}} \left(S - S_{34} - m_2^2 \right)$$

$$\mathcal{E}_4^* = \frac{1}{2\sqrt{S_{34}}} \left(S_{34} + m_4^2 - m_3^2 \right)$$

massless limit :

$s_{34} : 0 \to s$

$s_{24} : 0 \to s - s_{34}$

$$\int d\Phi_3 \to \frac{1}{32\pi^3} \frac{1}{4s} \int_0^s ds_{34} \int_0^{s-s_{34}} ds_{24}$$

$$= \frac{s}{256\pi^3} \quad //$$

We can also do this in energy space :

$$\mathcal{E}_2 \longrightarrow \frac{s + m_2^2 - s_{34}}{2\sqrt{s}}$$

$\mathcal{E}_2$ ranges from

obviously !

$$\mathcal{E}_2^{min} = \frac{s + m_2^2 - s_{34}^{max}}{2\sqrt{s}} = m_2$$

$$\mathcal{E}_2^{max} = \frac{s + m_2^2 - s_{34}^{min}}{2\sqrt{s}} = \frac{s + m_2^2 - (m_3 + m_4)^2}{2\sqrt{s}}$$

$\longrightarrow$ hard to extract the range of $\mathcal{E}_3$ in the presence of $\mathcal{E}_2$ ...

$\longrightarrow$ the trick :

consider $P_{23} \cdot P_1$

$$P_{23} \cdot P_1 = (E_2 + E_3)\sqrt{s} = \frac{1}{2}\left(s + P_{23}^2 - m_4^2\right)$$

where

$$P_{23} = (E_2 + E_3)^2 - \left(P_2^2 + P_3^2 - 2 P_2 P_3 Z\right)$$

$$= 2 E_2 E_3 + M_2^2 + M_3^2 + 2 P_2 P_3 Z$$

this summarizes to

$$E_3 \; 2(\sqrt{s} - E_2) +$$
$$2\sqrt{s}\, E_2 - s + m_4^2 - m_2^2 - m_3^2 = 2 P_2 P_3 Z$$

the limits of $E_3 \Rightarrow Z = \pm 1$

$$E_3 \; 2(\sqrt{s} - E_2) +$$
$$2\sqrt{s}\, E_2 - s + m_4^2 - m_2^2 - m_3^2 = \pm 2 P_2 P_3$$

solve this eq^n. for E_3

$$\sqrt{E_3^2 - m_3^2}$$

$$\sqrt{E_2^2 - M_2^2}$$

or

$$(LHS)^2 = (RHS)^2$$

& numerically solve for 2 roots

massless case

$$\varepsilon_2 : \quad 0 \longrightarrow \frac{\sqrt{s}}{2}$$

$$\varepsilon_3 : \quad \frac{\sqrt{s}}{2} - \varepsilon_2 \longrightarrow \frac{\sqrt{s}}{2}$$

$$\int d\Phi_3 = \int \frac{1}{32\pi^3} \, d\varepsilon_2 \, d\varepsilon_3 .$$

$$= \frac{1}{32\pi^3} \int_0^{\sqrt{s}/2} d\varepsilon_2 \quad \varepsilon_2$$

$$= \frac{1}{256\pi^3} s$$

//

agree w previous result

thus in energy space

$$\int d\Phi_3 \longrightarrow \frac{1}{32\pi^3}\int dE_2\, dE_3$$

E_2 : m_2 to $\dfrac{s+m_2^2-(m_3+m_4)^2}{2\sqrt{s}}$

E_3 : solving

$$E_3\; 2(\sqrt{s}-E_2)+$$

$$2\sqrt{s}\,E_2 - s + m_4^2 - m_2^2 - m_3^2 = \pm 2 p_2 p_3$$

massless case

E_2 : $0 \longrightarrow \dfrac{\sqrt{s}}{2}$

E_3 : $\dfrac{\sqrt{s}}{2} - E_2 \longrightarrow \dfrac{\sqrt{s}}{2}$.