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Fermi Golden Rule

①

Sakurai p. 341.

$$H = H_0 + V(t)$$

$$H_0 |n\rangle = E_n |n\rangle$$

$$|n(t)\rangle = e^{-iE_n t} |n\rangle$$

$\{|n(t)\rangle\}$ can span the state space at time t

$\Downarrow$

$$|\psi(t)\rangle = \sum_n c_n(t)\, e^{-iE_n t} |n\rangle$$

$$i\partial_t |\psi(t)\rangle = H |\psi(t)\rangle$$

$$\Rightarrow \sum_n \left(i\dot{c}_n + \cancel{E_n c_n}\right) e^{-iE_n t} |n\rangle =$$

$$\cancel{\sum_n c_n e^{-iE_n t} E_n |n\rangle} +$$

$$\sum_n c_n e^{-iE_n t} V |n\rangle$$

$$\sum_n i\dot{c}_n e^{-iE_n t} |n\rangle = \sum_n c_n e^{-iE_n t} V |n\rangle$$

or

$$\dot{c}_n = -i \sum_{n'} e^{i(E_n - E_{n'})t} c_{n'} \langle n|V|n'\rangle$$

$\langle n|V|n'\rangle \to V_{nn'}$

matrix form

$$\dot{\vec{C}} = -i \overleftrightarrow{V}_I \cdot \vec{c}$$

$$(V_I)_{ij} = \langle i| e^{iH_0 t} V e^{-iH_0 t} |j\rangle$$
$$= e^{i(E_i - E_j)t} \langle i|V|j\rangle.$$

leading order result:

$c_i(0) = 1 \qquad 0$ otherwise

$$\dot{c}_f \simeq -i V_{I\,fi} = -i e^{i(E_f - E_i)t} V_{fi}$$

$$c_f \to -i \int^t dt' e^{i(E_f - E_i)t'} V_{fi}$$

if we consider $V(t) \to \theta(t) V$

$$c_f \to -i V_{fi} \int_0^t dt' e^{i(E_f - E_i)t'}$$

$$= -i V_{fi} \frac{(e^{i(E_f - E_i)t} - 1)}{i(E_f - E_i)}$$

Prob. $\sim |c_f|^2$

$t \to \infty$

$\longrightarrow |V_{fi}|^2 \, 2\pi \, \delta(E_f - E_i) \, t$

to see this, either use

$$|c_f|^2 = |V_{fi}|^2 \frac{1}{(E_f - E_i)^2} \left| e^{i(E_f-E_i)t} - 1 \right|^2$$

$$= |V_{fi}|^2 \frac{1}{(E_f - E_i)^2} \, 4 \sin^2\left(\frac{E_f - E_i}{2} t\right)$$

$$\longrightarrow |V_{fi}|^2 \, 2\pi \, \delta(E_f - E_i) \, t$$

using

$$\lim_{\alpha \to \infty} \frac{\sin^2 \alpha x}{\alpha x^2} = \pi \, \delta(x)$$

Sakurai p. 343

(take $\alpha \sim t/2$)

or from definition.

$$c_f = -i \, V_{fi} \int_0^t dt' \, e^{i(E_f - E_i)t'}$$

$$|c_f|^2 \to |V_{fi}|^2 \int_0^t dt' \int_0^t dt'' \, e^{i(E_f - E_i)t'} e^{i(E_f - E_i)t''}$$

$$= |V_{fi}|^2 \int_0^t dt' \, e^{i(E_f - E_i)t'} \, 2\pi \, \delta(E_f - E_i)$$

$$= |V_{fi}|^2 \, 2\pi \, \delta(E_f - E_i) \, t \quad //$$

the total transition rate

$$\frac{d}{dt} \text{Prob.} \rightarrow \int |V_{fi}|^2 \, 2\pi \, \delta_{E_f - E_i} \, dN_f .$$

\# of final states

$$\rightarrow |V_{fi}|^2 \int dE_f \, 2\pi \delta(E_f - E_i) \, \frac{dN_f}{dE_f}$$

DOS.

$\rho(E_f)$.

$$\text{rate} = |V_{fi}|^2 \, 2\pi \, \rho(E_i).$$

$E_i = E_f$

Harmonic Perturbation.

if $V \rightarrow$ constant

$E_f \rightarrow E_i$ at large t

what if the interaction has a characteristic frequency ω s.t.

$$V(t) = V e^{i\omega t} + V^{+} e^{-i\omega t} \qquad t > 0.$$

$$c_f \to -i \int_0^t dt' \left[V_{fi}\, e^{i(E_f - E_i + \omega)t'} + V^{+}_{fi}\, e^{i(E_f - E_i - \omega)t'} \right]$$

$$= V_{fi} \frac{1}{E_f - E_i + \omega} \left(1 - e^{i(E_f - E_i + \omega)t}\right) +$$

$$V^{+}_{fi} \frac{1}{E_f - E_i - \omega} \; 1 - e^{i(E_f - E_i - \omega)t}$$

physics dictate only one term can be important for a given E_f.

$\to E_i - \omega$. emission

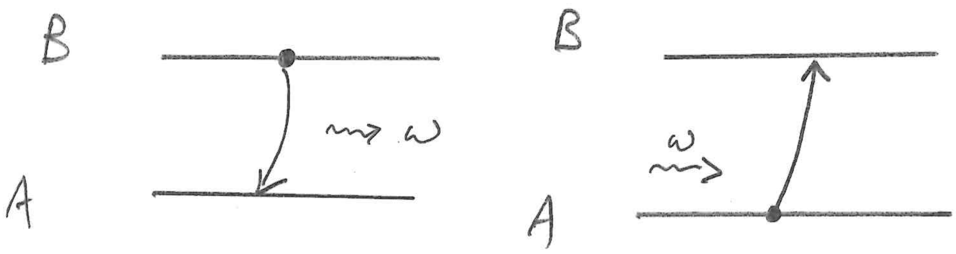
$\to E_i + \omega$ absorption.

$$|c_f|^2 \approx \begin{cases} |V_{fi}|^2 \, 2\pi \delta(E_f - E_i + \omega). \\ |V^{+}_{fi}|^2 \, 2\pi \delta(E_f - E_i - \omega). \end{cases}$$

Energy is not conserved.

$\to$ extract & release E from external source

Emission. Absorption



$E_A = E_B - \omega$

$E_B = E_A + \omega$

B is final state here

$\langle A | V | B \rangle$

$\langle B | V^\dagger | A \rangle$

$= \langle A | V | B \rangle^*$

$$|V_{AB}|^2 = |V^\dagger_{BA}|^2.$$

$\Downarrow$ consequence:

$$\frac{w^{\text{Emit}}_{B \to A + \omega}}{\rho_{E_A}} = \frac{w^{\text{Absorb.}}_{A + \omega \to B}}{\rho_{E_B}}$$

detailed balance!

Go beyond leading order

instead of $c_n^{(0)}(t) \approx \delta_{ni}$ in rhs

$$\Rightarrow c_n^{(1)}(t) = -i V_{ni} \int_0^t dt'' \, e^{i(E_n - E_i)t''}$$

$$\dot{c}_f \approx -i V_{fi} \, e^{i(E_f - E_i)t} +$$

$$-i \sum_{n'} e^{i(E_f - E_{n'})t} V_{fn'} \underbrace{c_{n'}^{(1)}(t)}$$

$$-i V_{n'i} \int_0^t dt'' \, e^{i(E_{n'} - E_i)t''}$$

$$= \frac{V_{n'i}}{E_i - E_{n'}} \left(e^{i(E_{n'} - E_i)t} - 1 \right)$$

neglect it when $\times \, e^{i(E_f - E_{n'})t}$ outside

$$\approx -i \left(V_{fi} + \sum_{n'} V_{fn'} \frac{1}{E_i - E_{n'}} V_{n'i} \right) \times e^{i(E_f - E_i)t}$$

$$\longrightarrow -i \, T_{fi} \, e^{i(E_f - E_i)t}$$

$$T = V + V \frac{1}{E_i - \hat{H}_0 + i\delta} T$$

LS eqn.

thus the Fermi Golden rule

$$d\Gamma = |T_{fi}|^2 \, 2\pi \, \delta_{E_f - E_i} \, dN_f(E_f).$$

Expert derivation

$$V(t) \sim \theta(t) V$$ not very natural

$$V(t) \rightarrow e^{\eta t} V$$

$$\eta > 0$$

kill the pot at $t \rightarrow -\infty$.

$$c_f^{(1)}(t) \rightarrow -i V_{fi} \int_{-\infty}^{t} dt' \, e^{i(E_f - E_i)t'} e^{\eta t'}$$

$$= \frac{V_{fi}}{E_i - E_f + i\eta} e^{i(E_f - E_i - i\eta)t}$$

$$|c_f^{(1)}|^2 = |V_{fi}|^2 \frac{e^{2\eta t}}{(E_i - E_f)^2 + \eta^2}$$

$$\frac{d}{dt}|c_f^{(1)}|^2 \longrightarrow |V_{fi}|^2 \frac{2\eta\, e^{2\eta t}}{(E_i - E_f)^2 + \eta^2}$$

$$\eta \to 0.$$

$$|V_{fi}|^2 \, 2\pi \, \delta(E_i - E_f)$$

$$\frac{1}{x + i\delta} = P\frac{1}{x} - i\pi\delta(x).$$

$$\mathrm{Im}\, \frac{1}{x - i\delta} = -\pi \delta(x)$$

$$2\pi\delta(x) \longleftrightarrow \frac{2\delta}{x^2 + \delta^2} \quad \delta \to 0$$

Same Golden Rule !

Now in the regulated world we can handle $E_f \to E_i$

$$C^{(0)}_{f\to i} = 1$$

$$C^{(1)}_{f\to i} = V_{ii} \frac{1}{i\eta} e^{\eta t}$$

$$C^{(2)}_{f} = \sum_{n'} \int_{-\infty}^{t} dt' \, e^{i(E_f - E_{n'} - i\eta)t'} \, (-iV_{fn'}) \times$$

$$\int_{-\infty}^{t'} dt'' \, e^{i(E_{n'} - E_i - i\eta)t''} \, (-iV_{n'i})$$

$$= \sum_{n'} \int_{-\infty}^{t} dt \, e^{i(E_f - E_i - 2i\eta)t} \frac{-V_{fn'}V_{n'i}}{i(E_{n'} - E_i - i\eta)}$$

$$C^{(2)}_{f\to i} \longrightarrow \sum_{n'} \frac{V_{fn'}V_{n'i}}{(E_f - E_i - 2i\eta)} \frac{1}{(E_{n'} - E_i - i\eta)} e^{i(E_f - E_i - 2\eta i)t}$$

$f \to i$

$n' = i$

$n' \neq i$

divergent !

$$= |V_{ii}|^2 \frac{-1}{2\eta^2} e^{2\eta t} +$$

$$-i \sum_{n' \neq i} |V_{in'}|^2 \frac{1}{2\eta(E_i - E_{n'} + i\eta)} e^{2\eta t}$$

$$c_{f\to i} \simeq 1 + -i V_{ii} \frac{1}{\eta} e^{\eta t}$$
$$+ (-i)^2 |V_{ii}|^2 \frac{1}{2\eta^2} e^{2\eta t}$$
$$-i \sum_{n'\neq i} |V_{in'}|^2 \frac{1}{2\eta\,(E_i - E_{n'} + i\eta)} e^{2\eta t}$$

is bad. qty to work w.

$$\dot{c}_{f\to i} \simeq -i V_{ii} + (-i)^2 \frac{1}{\eta} |V_{ii}|^2 e^{2\eta t}$$
$$-i \sum_{n'\neq i} |V_{in'}|^2 \frac{e^{2\eta t}}{E_i - E_{n'} + i\eta}.$$

$$\frac{\dot{c}_{f\to i}}{c_{f\to i}} = \frac{-i V_{ii} + (-i)^2 \frac{1}{\eta} |V_{ii}|^2 e^{2\eta t} - i \sum_{n'\neq i} |V_{in'}|^2 \frac{e^{2\eta t}}{E_i - E_{n'} + i\eta}}{1 - i V_{ii} \frac{1}{\eta} e^{\eta t} + \cdots}$$

$$\simeq -i V_{ii} - i \sum{}' |V_{in'}|^2 \frac{e^{2\eta t}}{E_i - E_{n'} + i\eta}$$

the div cancel.

$\frac{d}{dt} \ln C_{f\to i}$ is finite

$\Rightarrow \quad C^R_{f\to i} \to e^{-i\Delta t}$

is good stuff

$$\Delta(t) \simeq V_{ii} + \sum_{n'}{}' \frac{|V_{in'}|^2}{E_i - E_{n'} + i\eta}$$

no div

Δ is complex $\Rightarrow$ state becomes unstable due to interaction

Real : level shift

Im : width

$$\Delta \to \Delta_{re} + i\Delta_{im}$$

$$\Delta_{re} = V_{ii} + \sum{}' \mathcal{P} \frac{1}{E_i - E_{n'}} |V_{in'}|^2$$

$$\Delta_{im} = -\pi \sum_{n'}{}' |V_{in'}|^2 \delta(E_i - E_{n'})$$

we see

$$\sum_{n'} \gamma_{i\to n'} = -2\,\mathrm{Im}\,\Delta$$

$$C^R_{f\to i} = e^{-i\Delta t}$$

$$= e^{-i\Delta_{re} t}\, e^{\Delta_{im} t}$$

$$|C^R_{f\to i}|^2 = e^{2\Delta_{im} t}$$

prob. at i   loss

$$|C^R_{f\to i}|^2 + \sum_{n'} \gamma_{i\to n'}\, t$$

$$1 + 2\Delta_{im} t + \sum_{n'} \gamma_{i\to n'}\, t$$

$= 1$ to leading order ...

$$\gamma = -2\,\mathrm{Im}\,\Delta$$

$$\Delta E = \mathrm{Re}\,\Delta$$

Full solution. Merzbacher. p. 482.

$|\psi(t)\rangle$

$$|\psi_I(t)\rangle = e^{iH_0 t}|\psi(t)\rangle \longrightarrow \mathcal{U}_I(t)$$

$$= e^{iH_0 t} e^{-iHt}|\psi(0)\rangle$$

$$i\partial_t|\psi_I(t)\rangle = e^{iH_0 t} V e^{-iHt}|\psi(0)\rangle$$

$$= V_I |\psi_I(t)\rangle$$

$$\hookrightarrow V_I = e^{iH_0 t} V e^{-iH_0 t}$$

$$|\psi_I(t)\rangle = \mathcal{U}_I(t, t_0)|\psi_I(t_0)\rangle$$

$$i\partial_t \mathcal{U}_I = V_I \mathcal{U}_I$$

$$\mathcal{U}_I = T\left\{e^{-i\int^t V_I\, dt'}\right\}.$$

V_I at different t do not commute

Finally we identify

$$C_f(t) \longrightarrow \langle f|\mathcal{U}_I|i\rangle.$$