Maxwell's Equations & Differential Forms

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

using

$$\vec{E} = -\nabla A^0 - \partial_t \vec{A}$$
$$\vec{B} = \nabla \times \vec{A}$$

$$E^i = F_{0i}$$
$$B^i = -\frac{1}{2}\varepsilon_{ijk} F_{jk}$$

$$\Rightarrow \quad F_{0i} = -E_i = E^i$$
$$F_{ij} = \varepsilon_{ijk} B_k = -\varepsilon_{ijk} B^k$$

$$F_{\mu\nu} \longleftrightarrow \begin{array}{c|ccc} 0 & E^1 & E^2 & E^3 \\ \hline -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{array}$$

$F^{\mu\nu}$: flipping the sign of E

$$F^{\mu\nu} \longleftrightarrow \begin{array}{c|ccc} 0 & -E^1 & -E^2 & -E^3 \\ \hline E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{array}$$

No.

Date

to derive the Maxwell's eqns.

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\Downarrow$$

$$\partial_i F^{i0} = j^0 \qquad \Rightarrow \qquad \nabla \cdot \vec{E} = \rho$$

$$\partial_t F^{0i} + \partial_j F^{ji} = j^i \qquad \qquad -\partial_t \vec{E} + \nabla \times \vec{B} = \vec{j}$$

what about the other 2 ?

$\Rightarrow$ properties of $F_{\mu\nu}$: the fact that $F_{\mu\nu}$ is constructed based on a vector potential A_μ

$$B^k = -\frac{1}{2}\varepsilon_{kij} F^{ij}$$

$$\partial_k B^k = \underbrace{-\frac{1}{2}\varepsilon_{kij}}_{\text{anti-sym}} \big(\underbrace{\partial_k \partial^i}_{\text{sym}} A^j - \partial_k \partial^j A^i\big)$$

$$= 0$$

$$\nabla \cdot \vec{B} = 0$$

Absence of magnetic monopole

Also

$$E^k = F^{k0} = \partial^k A^0 - \partial^0 A^k$$

$$\varepsilon_{ijk}\,\partial_j E^k = \varepsilon_{ijk}\,\partial_j \partial^k A^0 - \varepsilon_{ijk}\,\partial_j \partial^0 A^k$$

$$= -\partial^0 (\nabla \times \vec{A})^i$$

$$\nabla \times \vec{E} + \partial_t \vec{B} = \vec{0}$$ Faraday's law

We see that the last 2 eqns are obtained quite unnaturally

⟶ we can do better

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\alpha F_{\mu\nu} = \partial_\alpha \partial_\mu A_\nu - \partial_\alpha \partial_\nu A_\mu$$

$$= \partial_\mu F_{\alpha\nu} + \partial_\mu \partial_\nu A_\alpha - \partial_\alpha \partial_\nu A_\mu$$

$$= \partial_\mu F_{\alpha\nu} + \partial_\nu F_{\mu\alpha}$$

or

$$\underline{\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu}} = 0$$

Bianchi identity

this summarizes the last 2 eqns

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

$$\Rightarrow \partial_i B_i = 0$$

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0$$

$$-\partial_t B^3 + \underbrace{\left(-\partial_1 E^2 + \partial_2 E^1\right)}_{-(\nabla \times \vec{E})^3} = 0$$

etc.

so the 4 Maxwell's eqns read

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0$$

note that

$$\partial_\nu j^\nu = \underbrace{\partial_\nu \partial_\mu}_{\text{sym.}} \underbrace{F^{\mu\nu}}_{\text{anti-sym}} = 0$$

current conservation

Finally observe the symmetry

$$\nabla \cdot \vec{E} = \rho \qquad \Big| \qquad \nabla \cdot \vec{B} = 0$$

$$-\partial_t \vec{E} + \nabla \times \vec{B} = \vec{J} \qquad \Big| \qquad -\partial_t \vec{B} - \nabla \times \vec{E} = \vec{0}$$

just

$\vec{E} \to \vec{B}$

$\vec{B} \to -\vec{E}$

& $J^\nu = 0$

When expressed in the language of differential form

$\Longrightarrow$ these relations will become more symmetric & elegant

No.

Date

we gain further insight with differential forms

// Differential Forms =

vector Calculus w/o vectors //

<u>1-form</u>

basis

$$\omega(1) = \omega_i \, dx^i$$

1-form from 0-form

$$ds = \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial y} dy + \frac{\partial s}{\partial z} dz$$

<u>2-form</u>

$$\omega(2) = \frac{1}{2!} \omega_{jk} \, dx^j \wedge dx^k$$

$$\updownarrow$$

$$\sim \varepsilon_{ijk} v_i$$

2-form can be created from 1-form.

No.

Date

$$\omega(1) = v_x\,dx + v_y\,dy + v_z\,dz$$

$$d\omega(1) = dv_x \wedge dx + dv_y \wedge dy + dv_z \wedge dz$$

$$= \left(\frac{\partial v_x}{\partial y} dy + \frac{\partial v_x}{\partial z} dz\right) \wedge dx + \cdots$$

$$\longrightarrow \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) dz \wedge dx$$

$$\sim (\nabla \times \vec{v}) \cdot d\vec{A}$$

<u>3-form</u>

$$\omega(3) = \frac{1}{3!}\, \omega_{ijk}\, dx^i \wedge dx^j \wedge dx^k$$

in 3D $\longrightarrow dx \wedge dy \wedge dz$ (1 term)

In the language of differential forms:

construct:

$$\overleftrightarrow{J} = j_\mu \, dx^\mu$$

$$\overleftrightarrow{F} = \frac{1}{2!} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

$$\overleftrightarrow{J} = j^0 \, dx^0 - j^1 \, dx^1 - j^2 \, dx^2 - j^3 \, dx^3$$

$$\overleftrightarrow{F} = E^1 dx^0 \wedge dx^1 + E^2 dx^0 \wedge dx^2 + E^3 dx^0 \wedge dx^3$$
$$- B^1 dx^2 \wedge dx^3 - B^2 dx^3 \wedge dx^1 - B^3 dx^1 \wedge dx^2$$

Construct the dual with Hodge Star

$$^*\overleftrightarrow{J} = \frac{1}{3!} \overset{*}{j}_{\alpha\beta\gamma} \, \underline{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}$$

$$\overset{*}{j}_{\alpha\beta\gamma} = j^\rho \, \varepsilon_{\rho\alpha\beta\gamma}$$

3-form

$\overleftrightarrow{J}$ & $^*\overleftrightarrow{J}$ represent the same info.

the independent entries are ...

$$^*J_{012} = -J^3$$

$$^*J_{031} = -J^2$$

$$^*J_{023} = J^1$$

same info as the 1-form

$$^*J_{123} = J^0$$

$$^*\overleftrightarrow{J} = J^0 dx^1 \wedge dx^2 \wedge dx^3$$
$$- J^1 dx^0 \wedge dx^2 \wedge dx^3$$
$$- J^2 dx^0 \wedge dx^3 \wedge dx^1$$
$$- J^3 dx^0 \wedge dx^1 \wedge dx^2$$

Similarly we can construct

$$^*\overleftrightarrow{F} = \frac{1}{2}\, ^*F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$^*F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$$
{}^{*}F_{\mu\nu} \longleftrightarrow
\begin{array}{c|ccc}
0 & -B^1 & -B^2 & -B^3 \\
\hline
B^1 & 0 & -E^3 & E^2 \\
B^2 & E^3 & 0 & -E^1 \\
B^3 & -E^2 & E^1 & 0
\end{array}
$$

just the same structure as

$F_{\mu\nu}$ but w

$$\vec{E} \to -\vec{B}$$
$$\vec{B} \to \vec{E}$$

we also see

$$\partial_\mu {}^{*}F^{\mu\nu} = \underline{0} \quad \longleftrightarrow \quad \begin{array}{l} \nabla\cdot\vec{B} = 0 \\ -\partial_t\vec{B} - \nabla\times\vec{E} = \vec{0} \end{array}$$

Vs

$$\partial_\mu F^{\mu\nu} = j^\nu$$

absence of magnetic monopole

$\Downarrow$

how differential forms provide more understanding ?

all relations are summarized as

① $\overleftrightarrow{F} = d\overleftrightarrow{A}$

$d\overleftrightarrow{F} = 0$

② $d{*}\overleftrightarrow{F} = -{*}\overleftrightarrow{J}$

$d{*}\overleftrightarrow{J} = 0$ $\qquad\qquad dd = 0$

let's unpack them

$$\overleftrightarrow{F} = d\overleftrightarrow{A}$$

$$= d\left(A_\mu dx^\mu\right)$$

$$= \frac{\partial A_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu$$

$$= \frac{1}{2} \underbrace{\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right)}_{F_{\mu\nu}} dx^\mu \wedge dx^\nu$$

it follows $\quad d\overleftrightarrow{F} = dd\overleftrightarrow{A} = 0$

Now $\quad d\overleftrightarrow{F} = \frac{1}{2}\left(dF_{\mu\nu}\right) \wedge dx^\mu \wedge dx^\nu$

$$= \frac{1}{2}\, \partial_\lambda F_{\mu\nu}\, dx^\lambda \wedge dx^\mu \wedge dx^\nu$$

$$= \frac{1}{3!}\left(\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\right) dx^\lambda \wedge dx^\mu \wedge dx^\nu$$

thus

$$d\overset{\leftrightarrow}{F} = 0 \iff \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

the other eq. reads

$$d\,{}^*\overset{\leftrightarrow}{F} = -{}^*\overset{\leftrightarrow}{J}$$

$$d\,{}^*\overset{\leftrightarrow}{F}$$

$$= d\left(\frac{1}{2}\,{}^*F_{\mu\nu}\,dx^\mu \wedge dx^\nu\right)$$

$$= \frac{1}{2}\left(d\,{}^*F_{\mu\nu} \wedge dx^\mu \wedge dx^\nu\right)$$

$$= \frac{1}{3!}\left(\partial_\lambda {}^*F_{\mu\nu} + \partial_\mu {}^*F_{\nu\lambda} + \partial_\nu {}^*F_{\lambda\mu}\right)$$

$$\underline{dx^\lambda \wedge dx^\mu \wedge dx^\nu}$$

a 3-form.

works it out :

$$-\nabla\cdot\vec{E}\quad dx^1 \wedge dx^2 \wedge dx^3 \;+$$

$$\left[-\partial_t E^1 + (\partial_2 B_3 - \partial_3 B_2)\right] dx^0 \wedge dx^2 \wedge dx^3 \;+$$

$$\left[-\partial_t E^2 + (\partial_3 B_1 - \partial_1 B_3)\right] dx^0 \wedge dx^3 \wedge dx^1 \;+$$

$$\left[-\partial_t E^3 + (\partial_1 B_2 - \partial_2 B_1)\right] dx^0 \wedge dx^1 \wedge dx^2$$

With Maxwell's eqns

$$
\begin{aligned}
d\,{}^{*}\overleftrightarrow{F} &= -j^0\, dx^1 \wedge dx^2 \wedge dx^3 \\
&\quad + j^1\, dx^0 \wedge dx^2 \wedge dx^3 \\
&\quad + j^2\, dx^0 \wedge dx^3 \wedge dx^1 \\
&\quad + j^3\, dx^0 \wedge dx^1 \wedge dx^2 \\
&= -{}^{*}\overleftrightarrow{j} \quad //
\end{aligned}
$$

see the expression of ${}^{*}\overleftrightarrow{j}$ derived before!

Now.

$$d\,d\,{}^{*}\overleftrightarrow{F} = 0 \quad \Rightarrow \quad -d\,{}^{*}\overleftrightarrow{j} = 0.$$

$$d\,{}^{*}\overleftrightarrow{j} = \underbrace{(\partial_\mu j^\mu)}_{0} \underbrace{(dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3)}_{\text{4-form.}}$$

current conservation is for free ...

Maxwell's eqns :

$$\mathcal{F} = dA \quad \text{(1)}$$

$$\Rightarrow d\mathcal{F} = 0 \quad \text{as} \quad dd = 0.$$

$$d{*}\mathcal{F} = -{*}\mathcal{J} \quad \text{(2)}$$

$$\Rightarrow d{*}\mathcal{J} = 0$$

No.

Date